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NUMERICAL METHODS FOR FUZZY INITIAL VALUE PROBLEMS

EYKE HÜLLERMEIER
IRIT, Université Paul Sabatier
F-31062 Toulouse, France
eyke@irit.fr

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In this paper, fuzzy initial value problems for modelling aspects of uncertainty in dynamical systems are introduced and interpreted from a probabilistic point of view. Due to the uncertainty incorporated in the model, the behavior of dynamical systems modelled in this way will generally not be unique. Rather, we obtain a large set of trajectories which are more or less compatible with the description of the system. We propose so-called *fuzzy reachable sets* for characterizing the (fuzzy) set of solutions to a fuzzy initial value problem. Loosely spoken, a fuzzy reachable set is defined as the (fuzzy) set of possible system states at a certain point of time, with given constraints concerning the initial system state and the system evolution. The main-part of the paper is devoted to the development of numerical methods for the approximation of such sets. Algorithms for precise as well as outer approximations are presented. It is shown that fuzzy reachable sets can be approximated to any degree of accuracy under certain assumptions. Our method is illustrated by means of an example from the field of economics.

Keywords: uncertain dynamics, fuzzy differential equations, fuzzy reachable sets, approximation, numerical computation.

1. Introduction

A common approach to the mathematical modelling of dynamical systems in engineering and the natural sciences is to characterize the system behavior by means of (ordinary) differential equations. Since these models are purely deterministic, the application of this approach requires precise knowlege about the system under investigation. However, this knowledge will rarely be available. On the contrary, parameter values, functional relationships, or initial conditions will often not be known precisely.

Consider, for instance, some physiological model in form of a set of differential equations. Some functional relationships between system variables are often known: "Right atrial pressure is equal to the mean circulatory filling pressure minus a pressure gradient which is the product of cardiac output and the resistance to venous return." However, several parameters of such equations may be unknown since system parameters are not directly measurable, those parameters may vary with time or differ from patient to patient, or exact values are only known for healthy patients but not under pathological conditions. Even worse, other functional relationships may only be characterized "linguistically" but not in terms of a precise mathematical model: "Increases in right atrial pressure produce increases in cardiac output only when right atrial pressure is relatively low and the ventricles are not distended" [4]. Despite these model uncertainties, it might be important to predict the system behavior, i.e., the patient's condition. For instance, it might be necessary to derive at least some bounds on the future cardiac output in order to decide on some medical treatment.

Probabilistic extensions of mathematical methods, such as stochastic differential equations [9], have been developed in order to take account of model uncertainties and the indeterminancy of the system evolution. However, recent research has shown that many different types of uncertainty exist [39] and that the framework of probability theory is perhaps inadequate to capture the full scope of uncertainty [20]. At least, alternative frameworks such as fuzzy set theory [38] and fuzzy measure theory [35] seem to complement probability theory in a meaningful way [19]. According to our opinion, this observation is also true for the domain of dynamical systems. Indeed, alternative approaches to modelling uncertain dynamical systems have been proposed recently in research areas such as *qualitative reasoning* [25,26] and fuzzy set theory [16,17,21,32].

The approaches just mentioned in connection with fuzzy set theory have proposed a concept of *fuzzy differential equations* which is principally based on the "fuzzification" of the differential operator. The definition of this operator makes use of a generalization of the so-called Hukuhara difference of sets $X, Y \subset \mathbb{R}^n$: A fuzzy set $\widetilde{Z} \in \mathbb{F}(\mathbb{R}^n)$ is called the H-difference of $\widetilde{X}$ and $\widetilde{Y}$, denoted $\widetilde{X} - \widetilde{Y}$, if $\widetilde{X} = \widetilde{Y} + \widetilde{Z}$. Here, $+$ is the usual addition of fuzzy sets and $\mathbb{F}(\mathbb{R}^n)$ denotes the set of all fuzzy subsets of $\mathbb{R}^n$. A fuzzy set $\widetilde{X}'(t_0)$ is defined to be the derivative of a fuzzy function $\widetilde{X} : \mathbb{R}^m \to \mathbb{F}(\mathbb{R}^n)$ at t_0 if the limits

$$\lim_{\Delta t \nearrow 0} \frac{\widetilde{X}(t_0 + \Delta t) - \widetilde{X}(t_0)}{\Delta t}, \quad \lim_{\Delta t \nearrow 0} \frac{\widetilde{X}(t_0) - \widetilde{X}(t_0 - \Delta t)}{\Delta t} \tag{1}$$

exist and are equal to $\widetilde{X}'(t_0)$. A solution to a *fuzzy initial value problem* (FIVP) based on this kind of derivative is thought of as a trajectory in the "state space" $\mathbb{F}(\mathbb{R}^n)$. However, the interpretation of such a solution seems unclear. At least, this approach leads to results which are not intuitive. Consider the (crisp) problem $\dot{x} = -x$, $x(0) \in [-1, 1]$ with an unknown initial system state as an example, where $\dot{x}$ denotes the derivative of the variable x. Since $x(t) = a\,\exp(-t)$ is the general solution of the initial value problem $\dot{x} = -x$, $x(0) = a$ and a is restricted to values within the interval $[-1, 1]$ in our example, we should expect to obtain a solution in form of a prediction $x(t) \in [-\exp(-t), \exp(-t)]$. However, the fuzzy function (which is actually a set-valued function) solving this initial value problem according to (1) is $\widetilde{X}(t) = [-\exp(t), \exp(t)]$. Particularly, we have $\mathrm{diam}(\widetilde{X}(t)) \to \infty$ instead of $\mathrm{diam}(\widetilde{X}(t)) \to 0$ as $t \to \infty$, where $\mathrm{diam}(X) := \sup\{|x - y| \mid x, y \in X\}$. Indeed, the fact that $\mathrm{diam}(\widetilde{X}(t))$ is non-decreasing in t, which means that the prediction of the system states $x(t)$ can only become less precise over time, for each solution of a fuzzy differential equation is a simple consequence of (1).

A solution to a FIVP based on (1), i.e., making use of generalized arithmetic operations and a fuzzy differential calculus [8], is interpreted as a single trajectory in $\mathbb{F}(\mathbb{R}^n)$. In [13] we have proposed a different definition of the "solution" to a FIVP which is closely related with set-valued analysis and set-valued differential equations [2,6]. Instead of one "fuzzy" trajectory, our interpretation is that of a (fuzzy) set of "ordinary" trajectories. According to our opinion, this approach produces more reasonable results. At least, it avoids such pathological examples as the one presented above. Moreover, as Section 2 will show, it can be well motivated from a semantical point of view.

In this paper, we are going to "operationalize" our approach., i.e., we are going to propose a method for computing approximations of (an adequate characterization of) the set of all solutions to a FIVP using numerical methods. Since finding this set of solutions analytically does only work with trivial examples, a numerical approach seems to be the only way of "solving" such problems.

The paper is organized as follows: For the sake of completeness we briefly review our approach to fuzzy differential equations* in Section 2, although most of the material of this section including technical details can also be found in the companion paper [13]. Numerical solution methods require a "discretization" of the original problem. In Section 3 several discretization steps in connection with fuzzy differential equations are discussed. Numerical approximation methods are then proposed in Section 4. A discussion of the results in Section 5 concludes the paper.

2. Fuzzy Initial Value Problems

2.1. *Modelling bounded uncertainty*

One way to model uncertainty in a dynamical system is to replace functions and initial values in the problem

$$\dot{x}(t) = f_0(t, x(t)), \qquad x(0) = x_0 \in \mathbb{R}^n \tag{2}$$

by set-valued functions and initial sets. This leads to the following (generalized) initial value problem (GIVP):

$$\dot{x}(t) \in F(t, x(t)), \qquad x(0) \in X_0 \subset \mathbb{R}^n \ , \tag{3}$$

where $F : [0, T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ is a set-valued function, X_0 is compact and convex. A solution x of (3) is understood to be an absolutely continuous function $x : [0, T] \to \mathbb{R}^n$ which satisfies (3) almost everywhere. The function F is taken to be set-valued in order to represent the bounded uncertainty of the dynamical system:

*As will become clear in subsequent sections, the notion *differential inclusion* seems more appropriate than *differential equation*. Nevertheless, we will use both terms interchangeably.

For each system state $(t, x) \in [0, T] \times \mathbb{R}^n$ the derivative is not known precisely, but it is an element of the set $F(t, x)$. With

$$\mathcal{X} := \{x : [0,T] \to \mathbb{R}^n \,|\, x \text{ is a solution of (3)}\} \subset C([0,T]) \ , \tag{4}$$

where $C([0,T])$ is the set of continuous functions on $[0,T]$, the reachable set $X(t)$ at time $t \in [0,T]$ is defined as

$$X(t) := \{x(t) \,|\, x \in \mathcal{X}\} \ .$$

The reachable set $X(t)$ is the set of all possible system states at time t. Knowledge about reachable sets is important for many applications such as, e.g., collision avoidance in robotics [11]. Thus, it seems meaningful to characterize the behavior of uncertain dynamical systems by means of reachable sets. Our objective is to find approximations of these sets for the system (3) using numerical methods. In order to define meaningful approximation procedures, it is necessary to know some properties of these sets. Results concerning such properties have been presented in [13]. Furthermore, we need a theoretical basis which allows us to solve (3) numerically, i.e., a discrete approximation of (3). Before we turn to these aspects, we consider a further generalization of (3).

2.2. *Fuzzy initial value problems*

A reasonable generalization of "set-valued" modelling, which takes aspects of gradedness into account, is the replacement of sets by fuzzy sets, i.e. (3) becomes the fuzzy initial value problem

$$\dot{x}(t) \in \widetilde{F}(t, x(t)), \quad x(0) \in \widetilde{X}_0 \tag{5}$$

on $J = [0,T]$ with a fuzzy function $\widetilde{F} : J \times \mathbb{R}^n \to \mathcal{E}^n$ and a fuzzy set $\widetilde{X}_0 \in \mathcal{E}^n$, where $\mathcal{E}^n$ is the set of normal, upper semicontinuous, fuzzy-convex, and compactly supported fuzzy sets $\widetilde{X} \in \mathbb{F}(\mathbb{R}^n)$. We suppose the right hand side to be continuous and bounded. Here, by continuity we mean continuity w.r.t the (generalized) Hausdorff metric on $\mathcal{E}^n$. The Hausdorff distance between two (nonempty) sets $A, B \subset \mathbb{R}^n$ is given as

$$d_H(A, B) := \max\{\beta(A,B), \beta(B,A)\} \ ,$$

where $\beta(A, B) := \sup_{x \in A} \rho(x, B)$ and $\rho(x, B) := \inf_{y \in B} |x - y|$. The generalization

$$\widetilde{d}_H(\widetilde{A}, \widetilde{B}) := \sup_{\alpha \in (0,1]} d_H([\widetilde{A}]_\alpha, [\widetilde{B}]_\alpha)$$

defines a distance measure for fuzzy sets $\widetilde{A}, \widetilde{B} \in \mathbb{F}(\mathbb{R}^n)$. The fuzzy function $\widetilde{F}$ is bounded if

$$s : J \times \mathbb{R}^n \to \mathbb{R}^n \ , \ (t, x) \mapsto \sup\{\, |y| \,|\, y \in \operatorname{supp}(\widetilde{F}(t,x))\}$$

is bounded, where $\text{supp}(\widetilde{X})$ is the support set of a fuzzy set $\widetilde{X}$.

For some $0 < \alpha \le 1$ consider the generalized initial value problem

$$\dot{x}(t) \in F_\alpha(t, x(t)), \quad x(0) \in [\widetilde{X}_0]_\alpha \ , \tag{6}$$

where F is defined pointwise as the α-cut $F_\alpha(t,z) := [\widetilde{F}(t,z)]_\alpha$ of the fuzzy set $\widetilde{F}(t,z)$. We call a function $x : J \to \mathbb{R}^n$ an α-solution to (5) if it is absolutely continuous and satisfies (6) almost everywhere on J. The set of all α-solutions to (5) is denoted $\mathcal{X}_\alpha$, and the α-reachable set $X_\alpha(t)$ is defined as

$$X_\alpha(t) := \{x(t) \,|\, x \in \mathcal{X}_\alpha\} \ .$$

Proposition 1 *Consider a* FIVP *and suppose the assumptions concerning the fuzzy right hand side and the initial set to hold. If the solution sets* $\mathcal{X}_\alpha$ $(0 < \alpha \le 1)$ *are interpreted as corresponding level-sets, the class* $\{\mathcal{X}_\alpha \,|\, 0 < \alpha \le 1\}$ *defines a normal fuzzy set* $\widetilde{\mathcal{X}} \in \mathbb{F}(C(J))$ *of solutions. Likewise, for all* $0 \le t \le T$*, the class* $\{X_\alpha(t) \,|\, 0 < \alpha \le 1\}$ *defines a normal fuzzy reachable set* $\widetilde{X}(t) \in \mathbb{F}(\mathbb{R}^n)$.

Proof. For $0 < \alpha < \beta \le 1$ we obviously have $\mathcal{X}_\beta \subset \mathcal{X}_\alpha$. In order to show that the class $\{\mathcal{X}_\alpha \,|\, 0 < \alpha \le 1\}$ defines a fuzzy set we still have to guarantee that (let $\mathcal{X}_0 := C(J)$)

$$\bigcap_{\alpha : \alpha < \beta} \mathcal{X}_\alpha = \mathcal{X}_\beta \tag{7}$$

for all $0 < \beta \le 1$. Observe that continuity and boundedness of $\widetilde{F}$ implies continuity and boundedness of F_α $(0 < \alpha \le 1)$. Moreover, since $\widetilde{F}$ is a fuzzy function and $\widetilde{X}_0$ is a fuzzy set, we have

$$\bigcap_{\alpha : \alpha < \beta} \text{graph}(F_\alpha) = \text{graph}(F_\beta) \quad \text{and} \quad \bigcap_{\alpha : \alpha < \beta} [\widetilde{X}_0]_\alpha = [\widetilde{X}_0]_\beta \ ,$$

where $\text{graph}(F) := \{((t,x), y) \,|\, y \in F(t,x)\}$. From $\widetilde{F}(t,x) \in \mathcal{E}^n$ also follows that F_α has compact and convex values. Then, (7) follows from Proposition 1 (Chapter 2.2) in [1]. As a consequence we also obtain

$$\bigcap_{\alpha : \alpha < \beta} X_\alpha(t) = X_\beta(t)$$

for all $0 \le t \le T$. Of course, $X_\beta(t) \subset X_\alpha(t)$ does also hold true for $0 < \alpha < \beta \le 1$, i.e., $\{X_\alpha(t) \,|\, 0 < \alpha \le 1\}$ defines a fuzzy reachable set.

Since $\widetilde{F}$ is continuous and normal on $J \times \mathbb{R}^n$, a continuous function $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying $f(t,x) \in F_1(t,x)$ on $J \times \mathbb{R}^n$ exists. Furthermore, $[\widetilde{X}_0]_1 \ne \emptyset$. Thus, a solution to (6) with $\alpha = 1$ exists (see basic existence theorems), and this solution belongs to $\widetilde{\mathcal{X}}_1$, which means that $\widetilde{\mathcal{X}}$ and $\widetilde{X}(t)$ are normal for all $0 \le t \le T$ □.

In connection with the numerical methods we are going to consider in subsequent chapters we will also assume that $\widetilde{F}$ satifies the Lipschitz condition

$$\tilde{d}_H(\widetilde{F}(t,x),\, \widetilde{F}(t,y)) \le L\,|x - y|$$

for all $t \in [0, T]$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with a Lipschitz constant $L > 0$.

2.3. *A probabilistic interpretation*

Our interpretation of a fuzzy initial value problem is that of a *generalized model of bounded uncertainty.* Such a model is thought of as a crude characterization of an underlying but not precisely known probabilistic model in form of a probability space $(\mathcal{C} \times D, \mathcal{A}, \mu)$. The probability space models the uncertainty concerning the unknown function $f_0 \in \mathcal{C}$ and the unknown initial system state $x_0 \in D$ in (2). Here, $\mathcal{C}$ is a certain class of functions, and D is a set of possible initial system states. The tuple $(f_0, x_0) \in \mathcal{C} \times D$ can be thought of as an unknown "parameter" of the initial value problem (2). In this case, the probability μ is interpreted (in a subjective sense) as a (conditional) distribution based on a body of knowledge, i.e., as a quantification of the belief concerning the "value" of (f_0, x_0).

The fuzzy right hand side in (5) is regarded as "weak" information about the probability μ: For certain values α the set-valued $(1-\alpha)$-section $F_{1-\alpha}$ of $\widetilde{F}$ together with the $(1-\alpha)$-cut $[\widetilde{X}_0]_{1-\alpha}$ are associated with an α-confidence region $\mathcal{C}_\alpha \times D_\alpha \subset \mathcal{C} \times D$ for the true but unknown tuple (f_0, x_0). More precisely, $\mathcal{C}_\alpha$ is defined as the set of all functions $f \in \mathcal{C}$ which are selections of $F_{1-\alpha}$, i.e., as the set of all functions $f \in \mathcal{C}$ satisfying $f(t, x) \in F_{1-\alpha}(t, x)$ on $J \times \mathbb{R}^n$. Likewise, D_α is defined as $[\widetilde{X}_0]_{1-\alpha} \subset \mathbb{R}^n$. Thus, we obtain $\text{Prob}\,((f_0, x_0) \in \mathcal{C}_\alpha \times D_\alpha) = 1 - \alpha$. It should be noted that an alternative interpretation of (f_0, x_0) as a random variable modelled by the probability space $(\mathcal{C} \times D, \mathcal{A}, \mu)$ is possible just as well.

Loosely spoken, the set $\mathcal{X}_\alpha$ of α-solutions is then thought of as an $(1 - \alpha)$-confidence region for the (unknown) solution(s) $x_0 : J \to \mathbb{R}^n$ to (2). Likewise, $X_\alpha(t)$ is a confidence region for the system state $x_0(t) \in \mathbb{R}^n$ at time t. The results obtained in [13] provide the formal basis for this interpretation. There, it is shown that the set $\mathcal{X}$ of solutions of a generalized initial value problem corresponds with the set of solutions associated with "ordinary" problems $\dot{x} = f(t, x)$, where f is a Carathéodory selection of F. Therefore, the class $\mathcal{C}$ should be defined as the class of all functions $f(t, x)$ measurable in t and continuous in x. Moreover, the existence of a probability measure compatible with the confidence regions associated with a fuzzy function $\widetilde{F}$ is proven.

3. Discretization of Fuzzy Initial Value Problems

3.1. *Approximation of fuzzy reachable sets*

The discussion in Section 2.2 made clear that the behavior of a fuzzy dynamical system, characterized by the "fuzzy funnel" $\{\widetilde{X}(t) \,|\, 0 \le t \le T\}$, can be described "levelwise" by the α-level sets $\{X_\alpha(t) \,|\, 0 \le t \le T\}$ for $\alpha \in (0, 1]$. For a certain value α, such a "crisp funnel" is defined by the reachable sets of a GIVP. Thus, we can characterize the set of solutions to a FIVP by characterizing the corresponding solution sets for a class of GIVPs.

Of course, it is not possible to compute the α-level sets for all values α within the interval $(0, 1]$. Therefore, our first approximation step is to characterize a fuzzy reachable set $\widetilde{X}(t)$ by a finite set of (crisp) reachable sets

$$\{X_\alpha(t) \mid \alpha \in A = \{\alpha_1, \ldots, \alpha_m\} \subset (0, 1]\}$$

and, hence, a "fuzzy funnel" by a finite set of "crisp funnels." The membership function of the fuzzy set $\widetilde{X}(t)$ is approximated by means of

$$\mu(x) = \max\{\alpha_k \mid x \in X_{\alpha_k}(t), 1 \leq k \leq m\} \ .$$

In [13] we have shown that this approximation makes sense: Under certain conditions, we can compute approximations to any degree of accuracy.

3.2. *Approximation of reachable sets*

After having reduced the problem of computing fuzzy reachable sets to one of computing "crisp" reachable sets, we now turn to the question of how to characterize such sets. For computing approximations of reachable sets $X(t)$ it is necessary to define a "discrete version" of (3). A simple first order discretization of (3) is given by

$$\frac{y_{i+1} - y_i}{\Delta t} \in F(t_i, y_i) \ , \tag{8}$$

where $0 = t_0 < t_1 < \ldots < t_N = T$ is a grid with stepsize $h = T/N = t_i - t_{i-1}$ $(i = 1, \ldots, N)$. As a solution to difference inclusion (8), we define any continuous and piecewise linear function $y^N : [0, T] \to \mathbb{R}^n$ such that

$$y^N(t) = y_i + \frac{1}{\Delta t}(t - t_i)(y_{i+1} - y_i) \qquad (t_i \leq t \leq t_{i+1}, i = 0, \ldots, N-1) \ ,$$

where $(y_0, \ldots, y_N)$ satisfies (8). Furthermore, let $\mathcal{S}^N$ denote the set of all such solutions (see [7]).

The following Euler scheme is the set-valued generalization of (8):

$$Y(t + \Delta t) = \bigcup_{y \in Y(t)} y + \Delta t F(t, y) \ . \tag{9}$$

Denote by Y_i^N the reachable set associated with (9) at time t_i, i.e.,

$$Y_i^N = \{x \in \mathbb{R}^n \mid x = y^N(t_i) \text{ for some } y^N \in \mathcal{S}^N\} \ .$$

Now, the question is whether $Y_N^N \to X(T)$ for $N \to \infty$.

A closed-valued and continuous function $R : [0, T] \to 2^{\mathbb{R}^n}$ is called an R-solution of the initial value problem

$$\dot{x}(t) \in F(t, x(t)) \qquad \text{almost everywhere on } [0, T], \qquad x(0) = x_0 \tag{10}$$

if $R(0) = \{x_0\}$ and

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} d_H(R(t+\Delta t), \bigcup_{x \in R(t)} x + \Delta t F(t,x)) = 0 \tag{11}$$

uniformly in $t \in [0,T]$ (see [31]). The union of the graphs of all solutions of (3) is usually called the *integral funnel.* Since the integral funnel is characterized by (11), this equation is also called *funnel equation* [31,33]. The following two theorems can be found in [31].

Theorem 1 *Suppose F to have nonempty, compact and convex values and to be continuous on $[0,T] \times \mathbb{R}^n$. Let the reachable set of problem (10) be bounded by $K \subset \mathbb{R}^n$. Furthermore, let F be Lipschitz in a neighborhood of K with respect to x, uniformly in $t \in [0,T]$. Then, a unique R-solution of (10) exists whose value at any $t \in [0,T]$ is the reachable set $X(t)$ of initial value problem (10).*

The next theorem shows that this unique R-solution is obtained by means of Y^N for $N \to \infty$ [30,31].

Theorem 2 *Under the conditions of Theorem 1,*

$$\max_{0 \le i \le N} d_H(Y_i^N, X(t_i)) \to 0 \qquad \text{as } N \to \infty \; .$$

For autonomous differential inclusions the following theorem is obtained in [37].

Theorem 3 *Let the set-valued function $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ have nonempty, compact and convex values. Moreover, let F be locally Lipschitz. Then, for the reachable sets $X(t)$ of the autonomous initial value problem (10) it holds true that $\lim_{N \to \infty} Y_N^N = X(T)$ for all $0 \le T < T(x_0)$. Here, the "escape time" $T(x_0)$ is defined as*

$$T(x_0) := \sup\{T \mid \bigcup_{0 \le t \le T} X(t) \text{ is compact }\} \; ,$$

i.e., $T(x_0)$ is the smallest T for which $|x(t)| \to \infty$ as $t \to T$ for some solution x of (10).

As we have seen, discrete methods can principally be used in order to find approximations of reachable sets $X(t)$ with any degree of accuracy. Based on a discretization (9), we can pass from the true solution $X(t)$ to an approximation $Y(t)$ such that $d_H(X(t), Y(t)) = O(\Delta t)$, i.e., the approximation error is at most linear in Δt. A further problem, however, is that of handling the approximating sets $Y(t_i) = Y_i^N$. Since these sets may have very complicated structures, it is generally not possible to represent them exactly. Thus, in addition to a discretization with regard to time, we have to perform an approximation of the class of all subsets $Y \subset \mathbb{R}^n$. For this purpose, consider a class $\mathcal{A} \subset 2^{\mathbb{R}^n}$ of sets which can be represented by means of a certain finite data structure. Moreover, for all $Y \subset \mathbb{R}^n$ suppose that a unique approximation $Z = \mathcal{A}(Y) \in \mathcal{A}$ does exist. We call the class $\mathcal{A}$ a δ-approximation of an (arbitrary) class $\mathcal{B} \subset 2^{\mathbb{R}^n}$ if

$$\forall Y \in \mathcal{B} : d_H(Y, \mathcal{A}(Y)) \le \delta \; .$$

Furthermore, we define the following approximation of (9):

$$Z_{n+1}^N = Z(t_{n+1}) = \mathcal{A}\left(\bigcup_{z\in Z_n^N} z + \Delta t\, \mathcal{A}(F(t_n,z))\right), \quad Z_0^N = \mathcal{A}(X_0)\ . \tag{12}$$

Proposition 2 *Suppose the class $\mathcal{A}$ to be a δ-approximation of the class of all subsets which have to be approximated in the iteration scheme (12). Moreover, suppose X_0 and $F(t,x)$ to be exacly approximated by $\mathcal{A}$, i.e., $X_0 \in \mathcal{A}$ and $F(t,x) \in \mathcal{A}$ for all $(t,x) \in [0,T]\times\mathbb{R}^n$. Then, $d_H(Y_n^N, Z_n^N) \le \delta T \exp(LT)/\Delta t$ for all $n \in \{0,\dots,N\}$.*

Proof. Consider Y_n^N and Z_n^N for some $n \in \{0,\dots,N-1\}$ and let the sets Y_{n+1}^N and V_{n+1}^N be defined as

$$Y_{n+1}^N = \bigcup_{y\in Y_n^N} y + \Delta t\, F(t_n,y), \quad V_{n+1}^N = \bigcup_{z\in Z_n^N} z + \Delta t\, F(t_n,z)\ .$$

For $y_1 \in Y_{n+1}^N$ we find some $y_0 \in Y_n^N$ and $\Delta y_0 \in F(t_n,y_0)$ such that $y_1 = y_0 + \Delta t\,\Delta y_0$. Let $z_0 \in Z_n^N$. Since $F(t_n,z_0)$ is compact, we find $\Delta z_0 \in F(t_n,z_0)$ such that $\rho(\Delta y_0, F(t_n,z_0)) = |\Delta y_0 - \Delta z_0|_2$. Thus, using the Lipschitz property of F,

$$|\Delta y_0 - \Delta z_0|_2 = \rho(\Delta y_0, F(t_n,z_0)) \le d_H(F(t_n,y_0),F(t_n,z_0)) \le L|y_0 - z_0|_2.$$

For $z_1 := z_0 + \Delta t \Delta z_0$ we have $z_1 \in V_{n+1}^N$ and

$$|y_1 - z_1|_2 \le |y_0 - z_0|_2 + \Delta t|\Delta y_0 - \Delta z_0|_2 \le (1+L\Delta t)|y_0 - z_0|_2.$$

Taking the infimum on the right-hand side (w.r.t. z_0) we obtain

$$|y_1 - z_1|_2 \le (1+L\Delta t)\rho(y_0, Z_n^N) \le (1+L\Delta t) d_H(Y_n^N, Z_n^N).$$

Since this holds for all $y_1 \in Y_n^N$ (and Y_n^N is compact,)

$$\beta(Y_{n+1}^N, V_{n+1}^N) \le (1+L\Delta t) d_H(Y_n^N, Z_n^N).$$

In the same way one verifies that $\beta(V_{n+1}^N, Y_{n+1}^N) \le (1+L\Delta t)d_H(Y_n^N, Z_n^N)$, which means $d_H(Y_{n+1}^N, V_{n+1}^N) \le (1+L\Delta t)d_H(Y_n^N, Z_n^N)$. Since $Z_{n+1}^N = \mathcal{A}(V_{n+1}^N)$, we have

$$\begin{aligned} d_H(Y_{n+1}^N, Z_{n+1}^N) &= d_H(Y_{n+1}^N, \mathcal{A}(V_{n+1}^N)) \\ &\le d_H(Y_{n+1}^N, V_{n+1}^N) + d_H(V_{n+1}^N, \mathcal{A}(V_{n+1}^N)) \\ &\le (1+L\Delta t)\, d_H(Y_n^N, Z_n^N) + \delta\ . \end{aligned} \tag{13}$$

The result then follows from the application of the (discrete) Gronwall lemma to (13) □.

Corollary 1 *Consider a sequence $(\Delta t_N) \subset \mathbb{R}_+$ with $\Delta t_N \to 0$ as $N \to \infty$. Moreover, suppose that for all $N \in \mathbb{N}$ there is a class $\mathcal{A}_N$ satisfying the assumptions in Proposition 2 for $\delta_N = \delta(\Delta t_N)$ and that $\delta_N = O((\Delta t_N)^2)$. Then,*

$$d_H(Z_n^N, X(n\,\Delta t)) \to 0 \quad as \quad N \to \infty\ .$$

Proof. We have $d_H(Z_n^N, X(n\,\Delta t)) \leq d_H(Z_n^N, Y_n^N) + d_H(Y_n^N, X(n\,\Delta t))$, and we already know that $d_H(Y_n^N, X(n\,\Delta t)) = O(\Delta t_N)$. Moreover, from Proposition 2 follows that

$$d_H(Z_n^N, Y_n^N) \leq \delta(\Delta t_N)\,T\,\exp(LT)/\Delta t_N = O(\Delta t_N)\;.$$

Therefore, $d_H(Z_n^N, X(n\,\Delta t)) = O(\Delta t_N)$ □.

Numerical methods for the simulation of ordinary initial value problems generally behave as follows: The smaller the step size Δt is chosen, the more precise the simulation results are. Interestingly enough, the same does not need to be true for the simulation of generalized initial value problems based on (12). On the one hand, smaller values Δt induce better approximations Y_n. On the other hand, the smaller Δt is, the more often such sets have to be approximated by sets $Z_n \in \mathcal{A}$. Since most of these approximations are afflicted with a corresponding approximation error, the overall result may become less precise. This is also the reason why δ_N has to be chosen of the order $O((\Delta t_N)^2)$ in Corollary 1.

4. Numerical Approximation Methods

The numerical methods presented in this section are based on the theoretical considerations of the previous sections, particularly on the discretization steps discussed there. The basic question we have to answer in the simulation, i.e., the numerical computation, of the behavior of fuzzy dynamical systems is the following: Given a fuzzy set $\widetilde{X}(t)$ of reachable system states at time t, what does the set $\widetilde{X}(t + \Delta t)$ look like? In Section 3 we have argued that the fuzzy trajectory associated with a fuzzy dynamical system can be approximated by a class of "crisp" funnels associated with certain generalized initial value problems.

A question we still have to answer is how to approximate (reachable) sets $X \subset \mathbb{R}^n$. There are different ways of representing such sets in finite-dimensional space, such as pointwise description, specification by their support functions or by means of a parametric description of their boundaries, or as level curves of smooth functions [22]. The latter aspect is closely related with efficient computation. Since a reachable set has to be represented by some (finite) data structure, it is not possible to deal with arbitrary sets $X \subset \mathbb{R}^n$. Therefore, X has to be replaced by some set Z which is an efficient representation from a computational point of view. As just mentioned, Z can be a parameterized set, that is, a set which is determined by a small number of parameters. Geometrical bodies such as rectangles, spheres and ellipsoids may serve as an example [27,28].

Intervals are a common approach for representing one-dimensional domains in numerical constraint satisfaction [5,29]. Relations $X \subset \mathbb{R}^n$ are then represented by their n one-dimensional projections, i.e. by an n-dimensional rectangle. This representation, which has also been used in simulation methods for uncertain dynamics [18,34], is very simple but it often leads to inaccurate results [12]. Therefore, we will make use of more complex approximations. To achieve this, we consider the class

$$\mathcal{A} = \mathcal{LCH} = \{\mathrm{lch}(A, \mathcal{H}) \mid A \subset \mathbb{R}^n,\ \mathrm{card}(A) < \infty,\ \mathcal{H} \subset 2^A\}$$

of *local convex hulls.*

Definition 1 ($\mathcal{LCH}$) *A* local convex hull *is defined as a hypergraph $G = (A, \mathcal{H})$, where $A \subset \mathbb{R}^n$,* $\mathrm{card}(A) < \infty$, *and $\mathcal{H}$ is a finite set of hyperedges $H \subset A$. The set of all local convex hulls is denoted $\mathcal{LCH}$ (where the dimension n is supposed to be fix.) The geometrical body $X \subset \mathbb{R}^n$ associated with the hypergraph G is defined as*

$$X := \bigcup_{H \in \mathcal{H}} \mathrm{conv}\, H$$

with $\mathrm{conv}\, H$ *the convex hull of the set H. X is also referred to as a local convex hull.*

The definition of a *discrete convex hull* in [10] can be seen as a special case of Definition 1 with the hyperedges H given implicitely. Based on a discretization of the state space into (rectangular) cells, a hyperedge is defined as a set of points located in a *neighborhood* of such cells.

Definition 2 (Δ-net) *Let $\Delta > 0$. A Δ-net $\mathcal{E}_\Delta$ in $\mathbb{R}^n$ is defined as*

$$\mathcal{E}_\Delta = \{[z_1\,\Delta, (z_1+1)\,\Delta] \times \ldots \times [z_n\,\Delta, (z_n+1)\,\Delta] \mid z_1, \ldots, z_n \in \mathbb{Z}\}\ .$$

For each point $x = (z_1\,\Delta, \ldots, z_n\,\Delta) \in \mathbb{R}^n$ with $z_1, \ldots, z_n \in \mathbb{Z}$ the union U of the set of 2^n "cells" $C \in \mathcal{E}_\Delta$ satisfying $x \in C$ is called a neighborhood. *The set of all neighborhoods in $\mathcal{E}_\Delta$ is denoted $\mathcal{N}(\mathcal{E}_\Delta)$.*

Definition 3 ($\mathcal{LCH}_\Delta$) *An* implicit local convex hull *or* discrete convex hull *associated with a Δ-net $\mathcal{E}_\Delta$ and a (finite) set $A \subset \mathbb{R}^n$ is defined as*

$$X := \bigcup_{U \in \mathcal{N}(\mathcal{E}_\Delta)} \mathrm{conv}(A \cap U)\ .$$

The set of all local convex hulls defined in this way is denoted $\mathcal{LCH}_\Delta$.

The approximation of sets $X \subset \mathbb{R}^n$ by the class $\mathcal{LCH}$ is very flexible. It combines advantages of global approximation with geometrical bodies (such as rectangles) and local approximation methods (such as the union of many small n-dimensional rectangles.) For example, with $\mathcal{H} = \{V\}$ we obtain the global approximation $\mathrm{lch}(G) = \mathrm{conv}\, V$, whereas $\mathcal{H} = \{\{x_1\}, \ldots, \{x_m\}\}$ leads to the local "approximation" $\mathrm{lch}(G) = V$.

4.1. *Precise approximations*

Consider a generalized initial value problem

$$\dot{x}(t) \in F(t, x(t)), \qquad x(0) \in X_0 \subset \mathbb{R}^n\ .$$

As far as properties of the function F and the initial set X_0 are concerned we have, according to our assumptions, that X_0 is convex and compact, F is bounded with nonempty, compact and convex values, and F is continuous (with respect to d_H). Moreover, F satisfies the Lipschitz condition $d_H(F(t,x),F(t,y)) \leq L\,|x-y|$ for all $x,y \in \mathbb{R}^n$ with a (global) Lipschitz constant $L > 0$. Particularly, we assume $X_0 \in \mathcal{A}$ and $F(t,x) \in \mathcal{A}$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$. By $\mathrm{lch}_\Delta(M)$ we denote the local convex hull defined implicitely by means of a Δ-net. For fixed $\Delta > 0$ let $\mathcal{A}_\Delta$ be the class of all such sets.

Definition 4 (δ-discretization) *Let $X \subset \mathbb{R}^n$. A finite set $M \subset X$ satisfying $d_H(M,X) \leq \delta$ is called a* δ-discretization *of X.*

The algorithm for computing approximations $Z_n = Z_n^N$ of the reachable sets $X(t_n)$ is based on the following iteration scheme:

$$Z_{n+1} = \mathrm{lch}_\Delta \left(\bigcup_{z \in \widehat{Z}_n} (z + \Delta t\, \widehat{F}(t_n,z)) \right), \quad Z_0 = X_0 \,. \tag{14}$$

Here, $\widehat{Z}_n$ is a δ-discretization of Z_n, and $\widehat{F}(t_n,z)$ is a δ-discretization of $F(t_n,z)$. Thus, each step of the iteration scheme consists of three main parts:

- *Discretization:* A discretization $\widehat{Z}_n$ of Z_n, as well as discretizations $\widehat{F}(t_n,z)$ of $F(t_n,z)$ have to be defined for all $z \in \widehat{Z}_n$.
- *Integration:* The set reachable at time t_{n+1} is characterized by a finite set of points, which is the union of the sets of points associated with each $z \in \widehat{Z}_n$ according to (14).
- *Approximation:* A new representation $Z_{n+1} \in \mathcal{A}_\Delta$ of the approximation of the reachable set $X(t_{n+1})$ has to be found.

Finding efficient algorithms for these parts, particularly the steps of discretization and integration, is a highly non-trivial task. We have developed and implemented corresponding algorithms [36] which are, however, not discussed in detail here.

We are now going to show that the algorithm based on (14) can compute approximations of reachable sets $X(t)$ to any degree of accuracy.

Lemma 1 *For all $M \subset \mathbb{R}^n$ and $\Delta > 0$ it holds true that $d_H(M, \mathrm{lch}_\Delta(M)) \leq \Delta$.*

Proof. Obviously, $M \subset \mathrm{lch}_\Delta(M)$, i.e., $\beta(M, \mathrm{lch}_\Delta(M)) = 0$. Now, consider some $x \in \mathrm{lch}_\Delta(M)$. Then, $x \in \mathrm{conv}\{x_1,\ldots,x_m\}$ for some points $x_1,\ldots,x_m$ located in an n-dimensional cube A with center $a = (a_1,\ldots,a_n) \in \mathbb{R}^n$ and length 2Δ. Now, suppose $|x - x_k|_\infty > \Delta$ for all $k \in \{1,\ldots,m\}$. W.l.o.g. suppose $x = (0,\ldots,0)$ and let $B = [-\Delta,\Delta]^n$. Then, $\{x_1,\ldots,x_m\} \cap B = \emptyset$. However, $C := B \cap \mathrm{conv}(A \setminus B)$ does not contain the origin in this case. This is obvious if $a_k = 0$ for some $1 \leq k \leq n$. Otherwise, some calculations show that C is the convex hull of a set of n rectangles of dimension $n-1$ which also does not contain the origin. However, this is a contradiction to the assumption that $x \in \mathrm{conv}\{x_1,\ldots,x_m\}$. Thus, there is some $k \in \{1,\ldots,m\}$ such that $x_k \in B$ and, hence, $|x - x_k|_\infty \leq \Delta$. Consequently, $\beta(\mathrm{lch}_\Delta(M), M) \leq \Delta$ and, therefore, $d_H(M, \mathrm{lch}_\Delta(M)) \leq \Delta$ $\square$.

Proposition 3 *Let Z_n be the approximation of the reachable set $X(t_n)$ obtained by means of (14). There are sequences $(\Delta_N) \subset \mathbb{R}_+$ and $(\delta_N) \subset \mathbb{R}_+$ so that $d_H(Z_n, X(t_n)) \to 0$ as $N \to \infty$ for all $n \in \{1, \dots, N\}$.*

Proof. First, observe that $\Delta t_N = T/N \to 0$ as $N \to \infty$. Now, for all $N \in \mathbb{N}$ we can choose Δ_N and δ_N in such way that $\Delta t_N \, \delta_N + (1 + \Delta t_N \, L) \, \delta_N + \Delta_N \le (\Delta t_N)^2$, where L is the Lipschitz constant associated with F. For some $n \in \{0, \dots, N-1\}$ consider the sets

$$A := \bigcup_{z \in Z_n} z + \Delta t \, F(t_n, z), \quad B := \bigcup_{z \in \widehat{Z}_n} z + \Delta t \, \widehat{F}(t_n, z) \ ,$$

where $\widehat{Z}_n$ is a δ-approximation of Z_n and $\widehat{F}(t_n, z)$ is a δ-approximation of $F(t_n, z)$. We have

$$d_H(A, \mathrm{lch}_\Delta(B)) \le d_H(A, B) + d_H(B, \mathrm{lch}_\Delta(B)) \le d_H(A, B) + \Delta$$

according to Lemma 1. Now, let us estimate $d_H(A, B)$. Since $B \subset A$, we have $d_H(A, B) = \beta(A, B)$. Consider some $x \in A$, i.e., $x = z + \Delta t \, \Delta z$, where $z \in Z_n$ and $\Delta z \in F(t_n, z)$. Since $\widehat{Z}_n$ is a δ-approximation of Z_n, there is a value $z' \in \widehat{Z}_n$ such that $|z - z'|_\infty \le \delta$. Since F is lipschitzean with Lipschitz constant L (and compact-valued,) we can find a value $\Delta z'' \in F(t_n, z')$ satisfying $|\Delta z - \Delta z''|_\infty \le L \, \delta$. Moreover, since $\widehat{F}(t_n, z')$ is a δ-approximation of $F(t_n, z')$, there is a $\Delta z' \in \widehat{F}(t_n, z')$ such that $|\Delta z' - \Delta z''|_\infty \le \delta$. For this value $\Delta z'$ let $y := z' + \Delta t \, \Delta z' \in B$. We obtain

$$\begin{aligned} |x - y|_\infty &= |z - z' + \Delta t(\Delta z - \Delta z')|_\infty \\ &\le |z - z'|_\infty + \Delta t \, |\Delta z - \Delta z'|_\infty \\ &\le \delta + \Delta t \, (|\Delta z - \Delta z''|_\infty + |\Delta z'' - \Delta z'|_\infty) \\ &\le \delta + \Delta t \, (L \, \delta + \delta) \ . \end{aligned}$$

Thus, $d_H(A, B) \le \Delta t \, \delta + (1 + \Delta t \, L) \, \delta$. Therefore, choosing for Δ and δ the values Δ_N and δ_N as indicated above, we obtain $d_H(A, \mathrm{lch}_{\Delta_N}(B)) \le (\Delta t_N)^2$. This means that $\mathcal{A}_{\Delta_N}$ satisfies the assumptions of Corollary 1 and, hence, $d_H(Z_n, X(n \, \Delta t)) \to 0$ as $N \to 0$ for all $n \in \{1, \dots, N\}$ $\square$.

Corollary 2 *Suppose the sequences $(\Delta_N) \subset \mathbb{R}_+$ and $(\delta_N) \subset \mathbb{R}_+$ to be defined as in Proposition 3 and $|F(t, x)| \le r$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$. If we extend the approximations Z_k of $X(t_k)$ to an approximation of $X(t)$ on $[0, T]$ by means of*

$$Z(t) = (t_{k+1} - t_k)/\Delta t \, Z_k + (t - t_k)/\Delta t \, Z_{k+1}$$

if $t_k \le t \le t_{k+1}$, then $Z(t) \to X(t)$ on $[0, T]$.

Proof. Let $T(k/N) = t_k \le t \le t_{k+1} = T((k+1)/N)$. Then,

$$d_H(Z_k, X(t)) \le d_H(Z_k, X(t_k)) + d_H(X(t_k), X(t)) \le d_H(Z_k, X(t_k)) + r \, \Delta t \ ,$$

i.e., $d_H(Z_k, X(t)) \to 0$ as $N \to \infty$ according to Proposition 3. In the same way one can show that $d_H(Z_{k+1}, X(t)) \to 0$ as $N \to \infty$. Therefore,

$$d_H(Z(t), X(t)) = d_H\left((t_{k+1} - t_k)/\Delta t\, Z_k + (t - t_k)/\Delta t\, Z_{k+1}, X(t)\right) \to 0$$

as $N \to \infty$ □.

4.2. *Outer approximations*

The computation of precise approximations of reachable sets may become very complex and time-consuming. In applications it is often not necessary to provide such precise results. Rather, one is often interested in estimations of reachable sets, particularly outer approximations $Z(t) \supset Y(t)$. Obviously, it is not difficult to define a "trivial" solution to this problem by simply making $Z(t)$ large enough. Since this does not make any sense, we will try to find a solution which satisfies $\beta(Y(t), Z(t)) = 0$ and which keeps $\beta(Z(t), Y(t))$ as small as possible. We are now going to consider a modification of our algorithm based on (14) for computing outer approximations of the sets $Y(t)$, i.e., the reachable sets associated with a discrete time scheme.

Lemma 2 *Let* $M = \{x_1, \dots, x_m\} \subset \mathbb{R}^m$, $m \geq 2$, $z \in \operatorname{conv} M$ *and suppose* $\operatorname{diam}(M) = \max\{|x - y| \mid x, y \in M\} \leq \delta$. *Moreover, suppose* F *to be lipschitzean with a Lipschitz-constant* L. *Then,*

$$\beta\left(z + \Delta t\, F(t,z), \operatorname{conv} \bigcup_{k=1}^{m} (x_k + \Delta t\, F(t, x_k))\right) \leq \Delta t\, L\, \delta\ .$$

Proof. We will prove this lemma for $m = 2$. The general case is treated analogously. Let $x, y, z \in \mathbb{R}^n$ and suppose $|x - y| \leq \delta$ and $z = \lambda x + (1 - \lambda) y$ for some $\lambda \in (0, 1)$. Let $A := x + \Delta t\, F(t, x)$, $B := y + \Delta t\, F(t, y)$, and $C := z + \Delta t\, F(t, z)$. Since A and B are convex, $\operatorname{conv}(A \cup B) = \bigcup_{0 \leq \lambda \leq 1} \lambda A + (1 - \lambda) B$, which means

$$\beta(C, \operatorname{conv}(A \cup B)) = \max_{z' \in C} \min_{x' \in A, y' \in B, \lambda \in [0,1]} |z' - (\lambda x' + (1 - \lambda) y')|\ .$$

Now, consider some $z' \in C$, i.e., $z' = z + \Delta t\, \Delta z$, where $\Delta z \in F(t, z)$, and recall that $d_H(F(t,x),F(t,z)) \le L|x-z|$ and $d_H(F(t,y),F(t,z)) \le L|y-z|$. Then,

$$\begin{aligned}
\rho(z',\operatorname{conv}(A\cup B)) &= \min_{x'\in A, y'\in B, \lambda\in[0,1]} |z+\Delta t\,\Delta z - (\lambda\, x' + (1-\lambda)\, y')| \\
&= \min_{\Delta x, \Delta y, \lambda} |z+\Delta t\Delta z - (\lambda x + (1-\lambda) y) - \Delta t(\lambda\,\Delta x + (1-\lambda)\,\Delta y)| \\
&\quad (\text{choose } \lambda \text{ such that } z = \lambda\, x + (1-\lambda)\, y) \\
&\le \min_{\Delta x, \Delta y} \Delta t\,|\lambda\,\Delta x + (1-\lambda)\,\Delta y - \Delta z| \\
&\le \min_{\Delta x, \Delta y} \Delta t\,(\lambda\,|\Delta x - \Delta z| + (1-\lambda)\,|\Delta y - \Delta z|) \\
&\le \Delta t\left(\lambda \min_{\Delta x}|\Delta x - \Delta z| + (1-\lambda)\min_{\Delta y}|\Delta y - \Delta z|\right) \\
&\le \Delta t\,(\lambda\, d_H(F(t,x),F(t,z)) + (1-\lambda)\, d_H(F(t,y),F(t,z))) \\
&\le \Delta t\,\max\{d_H(F(t,x),F(t,z)), d_H(F(t,y),F(t,z))\} \\
&\le \Delta t\, L\,\max\{|x-z|,|y-z|\} \\
&\le \Delta t\, L\,\delta.
\end{aligned}$$

Therefore, $\beta(C,\operatorname{conv}(A\cup B)) \le \Delta t\, L\,\delta$ □.

Lemma 3 *Let $\{x_1,\ldots,x_m\} \subset \mathbb{R}^m$, $\varepsilon > 0$ and denote by A_k the set of vertices of the cube $\{x \in \mathbb{R}^n \,|\, |x-x_k|_\infty \le \varepsilon\}$. Moreover, let $\overline{B}_\varepsilon(0) = \{x \in \mathbb{R}^n \,|\, |x|_\infty \le \varepsilon\}$. Then,*

$$\operatorname{conv}\{x_1,\ldots,x_m\} + \overline{B}_\varepsilon(0) \subset \operatorname{conv}(A_1 \cup \ldots \cup A_m)\ .$$

Proof. Consider some $y \in \operatorname{conv}\{x_1,\ldots,x_m\} + \overline{B}_\varepsilon(0)$. Then, $y = \sum_{k=1}^m \lambda_k x_k + a$, where $\lambda_k \ge 0$, $\sum_{k=1}^m \lambda_k = 1$, and $|a|_\infty \le \varepsilon$. That is,

$$y = \sum_{k=1}^m \lambda_k x_k + a = \sum_{k=1}^m \lambda_k (x_k + a) = \sum_{k=1}^m \lambda_k x'_k,$$

where $x_k \in \operatorname{conv} A_k$ $(1 \le k \le m)$. Therefore, $y \in \operatorname{conv}(A_1 \cup \ldots \cup A_m)$ □.

Lemma 4 *Consider a finite set $M \subset \mathbb{R}^n$ and suppose*

$$\operatorname{diam}(M) = \max\{|x-y|_\infty \,|\, x,y \in M\} \le \Delta\ .$$

Then, $\operatorname{conv} M \subset \operatorname{lch}_\Delta(M)$.

Proof. For some $x \in M$ we have $M \subset \operatorname{conv} M \subset B_\Delta(x)$. Obviously, $B_\Delta(x)$ is contained in some neighborhood of 2^n Δ-cells □.

Now, consider the following iteration scheme:

$$Z_{n+1} = \operatorname{lch}_\Delta\left(\bigcup_{z\in\widehat{Z}_n} W_{\Delta t L\delta}(z + \Delta t\,\widehat{F}(t_n,z))\right), \quad Z_0 = X_0\ . \tag{15}$$

Here, $\widehat{Z}_n$ is a δ-discretization of Z_n with following property:

$$\forall\, z \in Z_n \,\exists\, M \subset \widehat{Z}_n \,:\, z \in \operatorname{conv} M \wedge \operatorname{diam}(M) \leq \delta\,. \tag{16}$$

Such discretizations can easily be constructed. $\widehat{F}(t_n, z)$ is a discretization of $F(t_n, z)$ containing the extreme points of $F(t_n, z)$, i.e., $F(t_n, z) \subset \operatorname{conv} \widehat{F}(t_n, z)$. Moreover, for some $\varepsilon > 0$ and a finite set $M \subset \mathbb{R}^n$ the set $W_\varepsilon(M)$ is defined by replacing each point $x \in M$ by the set of vertices of the cube $\overline{B}_\varepsilon(x)$.

Proposition 4 *The algorithm based on iteration scheme (15) computes outer approximations $Z_n \supset Y_n$ of the reachable sets Y_n if $|F(t,x)|_\infty \leq r$ on $[0,T] \times \mathbb{R}^n$ and $\Delta \geq (1 + 3\,\Delta t\, L)\delta + \Delta t\, r$.*

Proof. Suppose that $Y_n \subset Z_n$ holds true for some $n \in \{0, \ldots, N-1\}$. We will show that this relation also holds true for Y_{n+1} and Z_{n+1}. Observe that

$$Y_n \subset Z_n \Rightarrow \bigcup_{y \in Y_n} y + \Delta t\, F(t_n, y) \subset \bigcup_{z \in Z_n} z + \Delta t\, F(t_n, z)\,.$$

Thus, it suffices to show that Z_{n+1} is an outer approximation of the second set, i.e., $z + \Delta t\, F(t_n, z) \in Z_{n+1}$ for all $z \in Z_n$. Let $z \in Z_n$. According to our assumption concerning the discretization $\widehat{Z}_n$ of Z_n, there is a set $M \subset \widehat{Z}_n$ such that $z \in \operatorname{conv} M$ and $\operatorname{diam}(M) \leq \delta$. From Lemma 2 and the fact that $F(t_n, x) \subset \operatorname{conv} \widehat{F}(t_n, x)$ follows that $\beta\,(z + \Delta t\, F(t_n, z), \operatorname{conv} M') \leq \Delta t\, L\, \delta$, where $M' = \bigcup_{x \in M}(x + \Delta t\, F(t_n, x))$. Now, consider the set $M'' = W_{\Delta t L \delta}(M')$. According to Lemma 3, we have $(z + \Delta t\, F(t_n, z)) \subset \operatorname{conv} M''$. For every two values $x, y \in M''$ there are values $x', y' \in M'$ such that $|x - x'|_\infty = |y - y'|_\infty = \Delta t\, L\, \delta$, i.e., $\operatorname{diam}(M'') \leq \operatorname{diam}(M') + 2\,\Delta t\, L\, \delta$. Moreover, for two points $x', y' \in M'$ we have

$$\begin{aligned} |x' - y'|_\infty &= |x + \Delta t\, \Delta x - y - \Delta t\, \Delta y|_\infty \\ &\leq \delta + \Delta t\, \rho(\Delta x, F(t_n, y)) + \operatorname{diam}(F(t_n, y)) \\ &\leq \delta + \Delta t\, (L\, \delta + r) \end{aligned}$$

with $x, y \in M$, $\Delta x \in F(t_n, x)$, $\Delta y \in F(t_n, y)$. Therefore, $\operatorname{diam}(M') \leq \delta + \Delta t\, (L\, \delta + r)$ and, hence, $\operatorname{diam}(M'') \leq ((1 + 3\,\Delta t\, L)\,\delta + \Delta t\, r)$. Thus,

$$z + \Delta t\, F(t_n, z) \subset \operatorname{conv} M'' \subset Z_{n+1}$$

according to the choice of Δ and Lemma 4 □.

4.3. *Examples*

Our first example discussed is based on a business cycle model which tries to explain how several causes of business activity induce business trends [24]. Under certain assumptions the dynamics of the corresponding economic system is described by the differential equations

$$\dot{x} = \alpha_x\, (\tanh(\kappa_x\, x + \sigma_x\, y) - x)\, \cosh(\kappa_x\, x + \sigma_x\, y) \tag{17}$$

$$\dot{y} = \alpha_y\, (\tanh(\kappa_y\, y + \sigma_y\, x) - y)\, \cosh(\kappa_y\, y + \sigma_y\, x)\,. \tag{18}$$

Here, the *state variables* $-1 \le x(t), y(t) \le 1$ characterize the stages of the business cycle. The parameters α_x, α_y, κ_x, κ_y, σ_x, σ_y, which have a certain economic interpretation do not only influence the quantitative but also the qualitative behavior of the system [23].

Now, suppose these parameters are not known precisely or are not constant over time. Rather, suppose the knowledge concerning a certain parameter to be given in form of a probability distribution resp. a system of confidence intervals. For example, suppose K_x to be an α-confidence interval for κ_x. This may be interpreted in different ways: (1) The (subjective) probability that the interval K_x covers the (fixed but unknown) parameter κ_x is α. (2) The probability that the (time-varying) parameter κ_x remains within the interval K_x for the complete time interval $[0, T]$ is α.

If the assumption of independence holds between the (random) parameters, we easily obtain a joint probability distribution, i.e., a system of confidence regions for the corresponding parameter vector $\theta = (\alpha_x, \alpha_y, \kappa_x, \kappa_y, \sigma_x, \sigma_y)$. From this distribution we can then derive a fuzzy function $\widetilde{F} : \mathbb{R}^2 \to \mathbb{F}(\mathbb{R}^2)$ characterizing the uncertain dynamic behavior of our economic system, i.e., (17) is replaced by

$$(\dot{x}, \dot{y}) \in \widetilde{F}(x, y) \ .$$

The values of an α-section F_α of $\widetilde{F}$ are given as

$$F_\alpha(x, y) = \{(f(x, y, \theta), g(x, y, \theta)) \in \mathbb{R}^2 \mid \theta \in C_\alpha\} \ ,$$

where the functions f and g are defined by the right hand sides of (17) and C_α is an α-confidence region for θ. Together with a fuzzy initial value (x_0, y_0) characterized by some fuzzy set $\widetilde{X}_0 \in \mathbb{F}(\mathbb{R}^2)$ we obtain a fuzzy initial value problem

$$(\dot{x}(t), \dot{y}(t)) \in \widetilde{F}(x(t), y(t)), \qquad (x(0), y(0)) \in \widetilde{X}_0$$

resp. a class of generalized inital value problems

$$(\dot{x}(t), \dot{y}(t)) \in F_\alpha(x(t), y(t)), \qquad (x(0), y(0)) \in [\widetilde{X}_0]_\alpha \ .$$

Simulation results for this system using the numerical methods discussed in the previous section are shown in Figure 1 and Figure 2. The uncertainty concerning four of the six parameters was modelled by means of fuzzy sets with (symmetric) triangular membership functions μ_{κ_x}, μ_{κ_y}, μ_{σ_x}, μ_{σ_y} with the support sets $(1.5, 1.6)$, $(1, 1.1)$, $(-0.5, -0.4)$, $(0.5, 0.6)$, respectively. Uncertainty concerning the initial system state was modelled by means of a "fuzzy circle" with radius 0.01 around the system state $(0.5, 0.5)$. Figure 1 shows the evolution of the α-cut of the fuzzy reachable set for $\alpha = 0.5$ and time points $t \in \{0, 1, \dots, 10\}$. The boundaries of these sets are depicted in the state space of the economic system. For instance, the small circle around the system state $(0.5, 0.5)$ is the 1/2-cut of the "fuzzy circle" modelling the uncertain system state at time $t = 0$. Likewise, the leftmost set is the 1/2-cut of the fuzzy reachable set at time $t = 10$. This set covers the true system

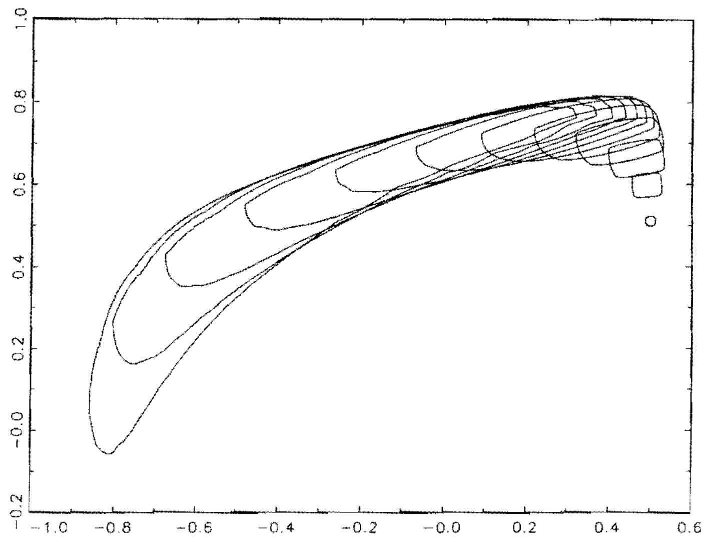

Figure 1: Simulation results for the dynamical economic system with imprecise parameters: Evolution of the α-cuts of the fuzzy reachable set for $\alpha = 0.5$.

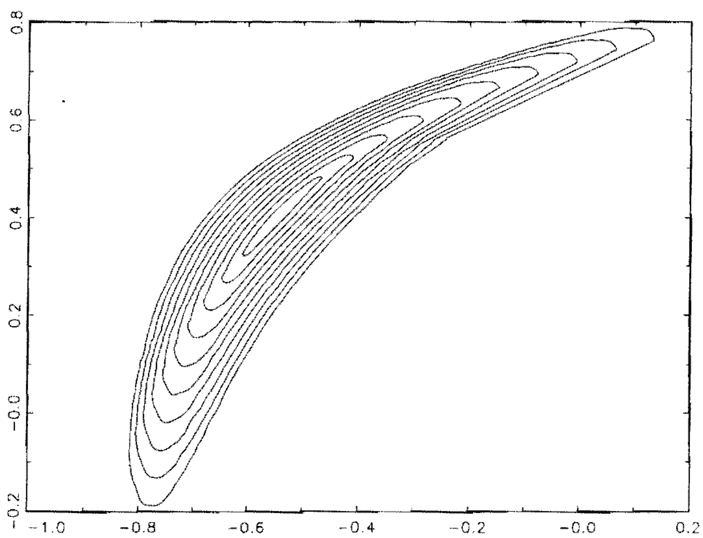

Figure 2: Simulation results for the dynamical economic system with imprecise parameters: Characterization of the fuzzy reachable set $\widetilde{X}(t)$ for $t = 11$.

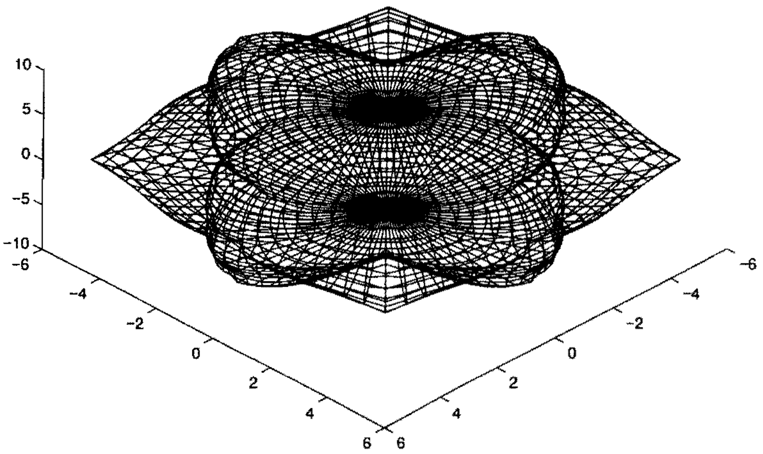

Figure 3: The approximation of the reachable set $X_{1/2}(5)$ of the system considered in the second example.

state $(x(10), y(10))$ with a probability of at least 1/2. A characterization of the fuzzy reachable set $\widetilde{X}(t)$ for $t = 11$ is shown in Figure 2, where the α-cuts of this set are depicted for $\alpha \in \{0.1, 0.2, \ldots, 1\}$. Notice that the sets represented by their boundaries in this picture correspond to different values of α, whereas those shown in Figure 1 correspond to different values of time t.

The second example is a generalization of an optimal control problem that has been studied in [3]. Consider the system

$$(\dot{x}, \dot{y}, \dot{z}) \in \varphi(|x|, |y|, |z|) \cdot \widetilde{W}, \qquad x(0) = y(0) = z(0) = 0 \ ,$$

where the fuzzy set $\widetilde{W}$ is characterized by the α-cuts

$$[\widetilde{W}]_\alpha = \{(a, b, c) \in \mathbb{R}^3 \mid |a|, |b|, |c| \leq 2\,(1 - \alpha)\} \ ,$$

and φ is defined by

$$\varphi(x, y, z) = 1 - \frac{\min\{|x|, |y|, |z|\}}{1 + \max\{|x|, |y|, |z|\}} \ .$$

Figure 3 shows the three-dimensional reachable set $X_\alpha(t)$ for $\alpha = 1/2$ and $t = 5$.

5. Discussion and Further Work

In this paper, we have outlined a method for modelling and simulation of uncertain dynamical systems. This approach is based on modelling uncertainty by means of fuzzy sets. Nevertheless, our interpretation of corresponding models is a probabilistic one. A "fuzzy" model is interpreted as a generalized model of bounded uncertainty, i.e., as a crude characterization of an underlying probabilistic model. In this sense, the approach establishes an interesting link between probability theory and the theory of fuzzy sets, which is closely related with *random sets* and *upper probabilities.*

The main-part of the paper is concerned with numerical methods for the computation of so-called fuzzy reachable sets. We have used such sets for characterizing the set of solutions to a fuzzy initial value problem. The methods, which were shown to provide outer approximations or even exact results in the limit, are based on three kinds of discretization: (1) Discretization with respect to *uncertainty* or *gradedness*, i.e., replacing the fuzzy problem by a finite number of crisp problems associated with certain α-sections of the fuzzy right hand side and the fuzzy initial system state. (2) Discretization with respect to *time*, i.e., computing reachable sets only for a discrete time scheme. (3) Discretization with respect to *space*, i.e., approximating reachable sets by geometrical bodies† and replacing such bodies by a finite number of points in $\mathbb{R}^n$.

Of course, the numerical simulation methods we have developed are very complex from a computational point of view. The reason for this complexity is the "pointwise" consideration of approximations of reachable sets in the discretization and integration phase of our numerical algorithm and the problem of handling (almost arbitrary) n-dimensional sets in the approximation phase of this algorithm. However, with regard to this point the following should be remarked: Firstly, we have shown elsewhere that very precise numerical results based on less complex methods can be found for special classes of systems, such as, e.g., quasi-monotone or linear systems [15]. Secondly, the precision of the results provided by our method is eligible. That is, less precise results can be obtained within reduced running time. Thirdly, parallel computation is a promising approach for coping with the complexity of approximation methods as defined in this paper. Since discretization and integration can be done simultaneously for different regions of a reachable set, these algorithms are well-suited for parallelization. The same is also true for the geometric approximation with local convex hulls. Therefore, an optimal speed-up is expected, and running time can be reduced considerably this way. Nevertheless, increasing the efficiency of simulation methods remains a central topic for future work. Particularly, developing specialized algorithms for certain classes of systems, i.e., exploiting the structure of such systems, seems to be a promising direction for further research.

The application of simulation methods for uncertain dynamics is another important aspect of future research. Such methods can be used in different ways. For example, they can be used directly in order to obtain predictions of the behavior of uncertain dynamical systems. The application of corresponding methods might be of particular interest in the so-called "soft sciences," where precise mathematical

†This is not really a discretization because the class of geometrical bodies we have considered is actually not discrete.

models are rarely available and probabilistic models are hard to specify. Moreover, the approach can also be used as a component of a certain reasoning mechanism such as, e.g., an expert system. For instance, our framework can be used for supporting the task of model-based diagnosis, where uncertain models appear due to the inherent vagueness or nonspecificity of (linguistic) hypotheses. More precisely, the simulation methods can be utilized as part of a methodology for testing hypotheses about dynamical systems. In [14] we have applied this approach for testing hypotheses about the misbehavior of a certain biological system.

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