Lossless compression

- $A = \{a_1, \ldots, a_d\}$ finite alphabet, $p = (p_1, \ldots, p_d) \in S^d$, i.e. probability distribution
- $X = X_1 \cdots X_l \in A^*$
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Want function $f : A \to \{0,1\}^*$ such that

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1. guarantees that X can be recovered from $f(X) = f(X_1) \cdots f(X_l)$
2. $E[f]$ called expected codeword length of f

Shannon code

Given $A = \{a_1, \ldots, a_d\}$, Shannon code $S : A \to \{0,1\}^*$ achieves

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Loss in compression: $E[S'] - E[S] = \sum p_i \log(p_i/q_i) = D_{KLD}(p, q)$.

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Loss in compression

Lemma 3.1

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Observation

$\forall x \in \mathbb{R}_+ : \ln(x) \leq x - 1.$

Markov models

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$\Rightarrow$ may outweigh gain of compression

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Goal Find centroids and corresponding partition that minimize loss in compression.

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$\Rightarrow$ k-median problem for Kullback-Leibler divergence

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