# VI. Theoretical constructions of pseudorandom objects

**Goal** pseudorandom generators and pseudorandom functions from general assumptions.

**Assumption** one-way functions/permutations exist.

one-way fcts/perm ➔ hardcore predicates

➔ PRG with expansion n+1

➔ PRG with polynomial expansion factor

➔ PRF

# Inverting game

$f:\{0,1\}^* \rightarrow \{0,1\}^*$, A a probabilistic polynomial time algorithm

**Inverting game $\mathrm{Invert}_{A,f}(n)$**

1. $x \leftarrow \{0,1\}^n, y := f(x)$.
2. A given input $1^n$ and y, outputs $x'$.
3. Output of game is 1, if $f(x') = y$, otherwise output is 0.

Write $\mathrm{Invert}_{A,f}(n) = 1$, if output is 1. Say A has succeded or A has won.

# Definition of one-way function

**Definition 6.1** $f:\{0,1\}^* \to \{0,1\}^*$ called one-way, if

1. there is a ppt $M_f$ with $M_f(x) = f(x)$ for all $x \in \{0,1\}^*$
2. for every probabilistic polynomial time algorithm A there is a negligible function $\mu:\mathbb{N} \to \mathbb{R}^+$ such that $\Pr\left[\text{Invert}_{A,f}(n) = 1\right] \leq \mu(n)$.

**Notation** $\Pr_{x \leftarrow \{0,1\}^n}\left[A(f(x)) \in f^{-1}(f(x))\right] \leq \mu(n)$

# Definition of one-way permutation

$f:\{0,1\}^* \to \{0,1\}^*$ length preserving, if for all x $|f(x)| = |x|$.

$f_n := f_{|\{0,1\}^n}$, restriction of f to $\{0,1\}^n$.

**Definition 6.2** A one-way function $f:\{0,1\}^* \to \{0,1\}^*$ is called one-way permutation, if

1. $f$ is length-preserving,
2. for every $n \in \mathbb{N}$ the function $f_n$ is a bijection.

# Function families

**Definition 6.3** A triple $\Pi = (\text{Gen}, \text{Samp}, f)$ of ppts is called a family of functions, if

1. $\text{Gen}(1^n)$ outputs parameters I with $|I| \geq n$, where each I defines finite sets $D_I$ and $R_I$ for a function $f_I : D_I \rightarrow R_I$ defined below.

2. $\text{Samp}(I)$ outputs $x \leftarrow D_I$.

3. f is deterministic and on input I, $x \in D_I$ outputs $y \in R_I$, $y := f_I(x)$.

$\Pi$ is a family of permutations, if in addition for all I $D_I = R_I$ and $f_I$ is a bijection

# The inverting games

**Inverting game $\mathrm{Invert}_{A,\Pi}(n)$**

1. $I \leftarrow \mathrm{Gen}(1^n), x \leftarrow \mathrm{Samp}(I), y := f_I(x)$.
2. A given input $1^n, I$ and y, outputs $x'$.
3. Output of game is 1, if $f_I(x') = y$, otherwise output is 0.

**Definition 6.4** A family of functions $\Pi = (\mathrm{gen}, \mathrm{Samp}, f)$ is called one-way, if for every probabilistic polynomial time algorithm A there is a negligible function $\mu : \mathbb{N} \rightarrow \mathbb{R}^+$ such that $\Pr[\mathrm{Invert}_{A,\Pi}(n) = 1] \leq \mu(n)$.

# Candidates

1. $$f_{mult}: \{0,1\}^* \to \{0,1\}^*$$
$$x \mapsto (x_1 \cdot x_2, |x_1|, |x_2|),$$

where $|x_1| = \lfloor |x|/2 \rfloor, |x_2| = \lceil |x|/2 \rceil$, and identify bit strings and integers via binary representations.

**Idea** Multiplication easy, factoring hard

# Candidates

2. $\text{Gen}(1^n)$ generates n n-bit integers uniformly at random,

$I = (a_1, \ldots, a_n)$

$\text{Samp}(I)$ $x \leftarrow \{0,1\}^n, x = (x_1, \ldots, x_n)$

$f_I(x)$ outputs $\sum_{i=1}^{n} x_i a_i$

**Idea** Addition is easy, SubsetSum is difficult.

# Candidates

3. $\mathrm{Gen}(1^n)$ generates prime number $p \geq 2^n$ and generator g for the multiplicative group $\mathbb{Z}_p^*$, $I = (p, g)$, $D_I = \mathbb{Z}_{p-1}$, $R_I = \mathbb{Z}_p^*$

   $\mathrm{Samp}(I)$ $x \leftarrow \mathbb{Z}_{p-1}$

   $f_I(x)$ outputs $g^x \mod p$

Idea Exponentiation is easy, discrete logarithm is difficult.

# Hardcore predicates

**Definition 6.5** $hc:\{0,1\}^* \to \{0,1\}$ is a hardcore predicate for a function $f:\{0,1\}^* \to \{0,1\}^*$, if

1. hc can be computed in polynomial time,
2. for every probabilistic polynomial time algorithm A there is a negligible function $\mu:\mathbb{N} \to \mathbb{R}^+$ such that
$\Pr_{x \leftarrow \{0,1\}^n}\left[A(f(x)) = hc(x)\right] \le 1/2 + \mu(n)$.

# The Goldreich-Levin predicate

$f:\{0,1\}^* \to \{0,1\}^*$ one-way, then

$$\begin{aligned} g: \quad \{0,1\}^* &\to \{0,1\}^* \\ w &\mapsto f(x) \| r, \text{ where } w = x \| r, |x| = |r|, \end{aligned}$$

is also one-way.

Formally, g is only definied for arguments of even length, by padding w we can define it for all bit strings.

**Theorem 6.6** Let f be a one-way function and g be defined as above. Then

$$\begin{aligned} gl: \quad \{0,1\}^* &\to \{0,1\} \\ (x,r) &\mapsto x \odot r = \sum x_i r_i \bmod 2 \end{aligned}$$

is a hardcore predicate for g.

# The Goldreich-Levin predicate

**Theorem 6.6 (reformulated)** Let f be a one-way function. Let g and the predicate gl be defined as above. If there exists a ppt A and a polynomial $p(\cdot)$ such that

$$\Pr_{x,r\leftarrow\{0,1\}^n}\left[A(f(x),r)=gl(x,r)\right]\geq\frac{1}{2}+\frac{1}{p(n)}$$

for infinitely many values of n, then there exists a ppt $A'$ and a polynomial $q(\cdot)$ such that

$$\Pr_{x\leftarrow\{0,1\}^n}\left[Invert_{A',f}(n)=1\right]\geq\frac{1}{q(n)}$$

for infinitely many values of n.

# An extremely simplified variant

**Theorem 6.7** Let f be a one-way function and let gl be the Goldreich-Levin predicate. If there exists a ppt A such that

$$\Pr_{x,r\leftarrow\{0,1\}^n}\left(A\left(f\left(x\right),r\right)=gl\left(x,r\right)\right)=1$$

for infinitely many values of n, then there exists a ppt A' such that

$$\Pr\left(\mathrm{Invert}_{A',f}\left(n\right)=1\right)=1$$

for infinitely many values of n.

# A simplified variant

**Theorem 6.8** Let f be a one-way function and let gl be the Goldreich-Levin predicate. If there exists a ppt A and a polynomial p such that

$$\Pr_{x,r\leftarrow\{0,1\}^n}\left(A\left(f\left(x\right),r\right)=gl\left(x,r\right)\right)\geq\frac{3}{4}+\frac{1}{p\left(n\right)}$$

for infinitely many values of n, then there exists a ppt A' such that

$$\Pr\left(\mathrm{Invert}_{A',f}\left(n\right)=1\right)\geq\frac{1}{4p\left(n\right)}$$

for infinitely many values of n.

# One-way functions and hard-core predicates

**Claim 6.9** Let f,gl, A, p be as before. Then there exists a set $S_n \subseteq \{0,1\}^n$ of size at least $\frac{2^n}{2p(n)}$ such that for every $x \in S_n$

$$\Pr_{r \leftarrow \{0,1\}^n}\left(A\left(f\left(x\right),r\right) = gl\left(x,r\right)\right) \geq \frac{3}{4} + \frac{1}{2p(n)}.$$

# One-way functions and hard-core predicates

**Claim 6.10** Let f,gl, A, p be as before. Then there exists a set $S_n \subseteq \{0,1\}^n$ of size at least $\frac{2^n}{2p(n)}$ such that for every $x \in S_n$ and every $i \in \{1,\ldots,n\}$

$$\Pr_{r \leftarrow \{0,1\}^n}\left(A\left(f\left(x\right),r\right) = gl\left(x,r\right) \wedge A\left(f\left(x\right),r \oplus e^i\right) = gl\left(x,r \oplus e^i\right)\right)$$

$$\geq \frac{1}{2} + \frac{1}{p(n)}.$$

# Chebyshev's inequality

**Theorem 6.11 (Chebyshev)** Let X be a random variable and $\delta > 0$. Then

$$\Pr\left[\left|X - E[X]\right| \geq \delta\right] \leq \frac{\mathrm{Var}[X]}{\delta^2}.$$

**Corollary 6.12** Let $X_1,\ldots,X_m$ be pairwise independent random variables with the same expectation $\mu$ and the same variance $\sigma^2$. Then, for every $\epsilon > 0$,

$$\Pr\left[\left|\frac{\sum_{i=1}^{m} X_i}{m} - \mu\right| \geq \epsilon\right] \leq \frac{\sigma^2}{\epsilon^2 m}.$$

# From prediction to inversion

1. For $i=1$ to $n$ do
2. For $j=2$ to $np(n)^2/2$ do
3. $r \leftarrow \{0,1\}^n$
4. $\bar{x}_{i,j} \leftarrow A(f(x),r) \oplus A(f(x),r \oplus e^i)$
5. $x_i := \text{majority}\left(\bar{x}_{i,1},\ldots,\bar{x}_{i,np(n)^2/2}\right)$
6. Output $x := x_1 \ldots x_n$

# Hardcore predicates and PRGs

**Theorem 6.13** Let f be a one-way permutation and hc a hardcore predicate for f. Then

$$\begin{aligned} G: \quad \{0,1\}^* &\rightarrow \{0,1\}^* \\ s &\mapsto f(s) \,\|\, hc(s) \end{aligned}$$

is a PRG with expansion factor $n+1$.

# Pseudorandom generators

**Definition 2.5 (restated)** Let $l : \mathbb{N} \to \mathbb{N}$ be a polynomial with $l(n) > n$ for all $n \in \mathbb{N}$. A deterministic polynomial time algorithm G is a pseudorandom generator if

1. $\forall s \in \{0,1\}^* \quad |G(s)| = l(|s|)$,
2. For every ppt D there is a negligible function $\mu : \mathbb{N} \to \mathbb{R}^+$ such that $\forall n \in \mathbb{N} \left| \Pr[D(r)=1] - \Pr[D(G(s))=1] \right| \le \mu(n)$, where $r \leftarrow \{0,1\}^{l(n)}$ and $s \leftarrow \{0,1\}^n$.

l is called the expansion factor of G.

# From distinguishers to predictors

**A on input $f(s)$**

1. $r' \leftarrow \{0,1\}$
2. Invoke D with input $f(s) \| r'$
3. If D returns 1, then output $r'$, otherwise output complement of $r'$.

# PRGs with arbitrary expansion

**Theorem 6.14** If there is a PRG $\bar{G}$ with expansion factor n+1, then there is a PRG G with expansion factor $p(n)$ for every polynomial $p:\mathbb{N} \to \mathbb{N}$ with $p(n) \geq n$ for all $n \in \mathbb{N}$.

# The construction

$\bar{G}$ PRG with expansion $n+1$

$G$ on input $s \in \{0,1\}^n$

1. $q(n) := p(n) - n.$
2. Set $s_0 := s, \sigma_0 := \varepsilon$ (empty string)
3. For $i = 1, \ldots, q(n)$ do:
   a) Define $s'_{i-1}$ to be the first $n$ bits of $s_{i-1}$
   and $\sigma_{i-1}$ to be the last $i-1$ bits of $s_{i-1}$.
   b) Set $s_i := \bar{G}(s'_{i-1}) \| \sigma_{i-1}$.
4. Output $s_{q(n)}$.

# The construction

G on input $s \in \{0,1\}^n$

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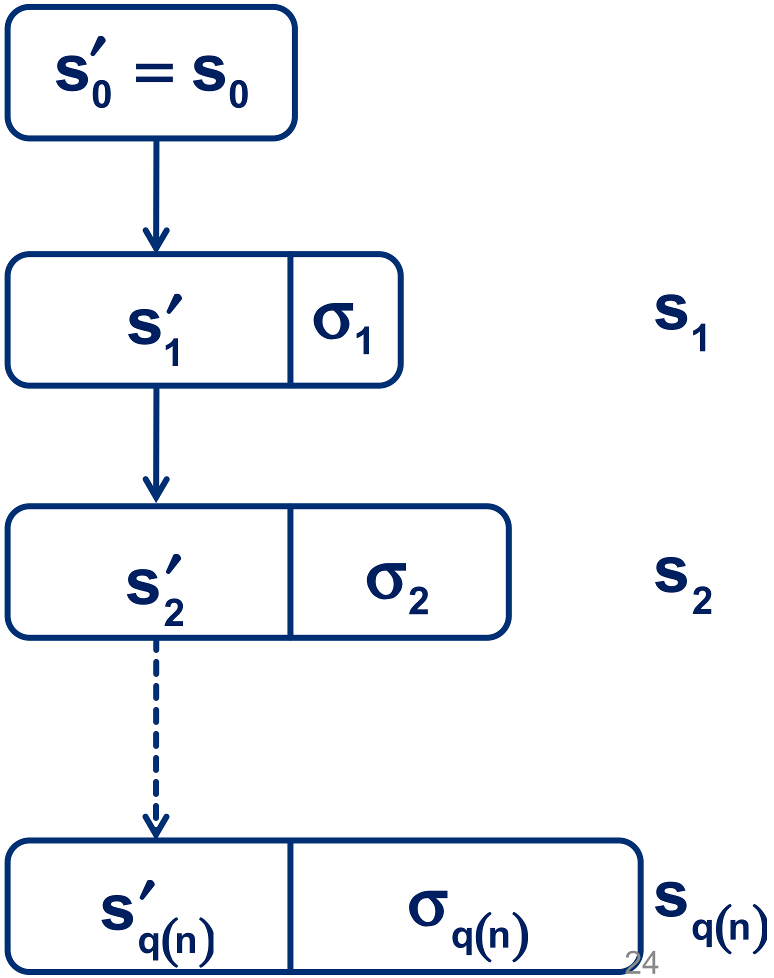
   b) Set $s_i := \bar{G}(s'_{i-1}) \| \sigma_{i-1}$.
4. Output $s_{q(n)}$.



# The construction –a special case

- $f : \{0,1\}^* \to \{0,1\}^*$ a one-way function
- $hc : \{0,1\}^* \to \{0,1\}$ a hardcore predicate for f.
- $\bar{G}(s) = f(s) \| hc(s)$

G with expansion factor $p(n)$:

$$G(s) = f^{(p(n)-n)}(s) \| hc\left(f^{(p(n)-n-1)}(s)\right) \| \cdots \| hc\left(f^{(1)}(s)\right) \| hc\left(f^{(0)}(s)\right)$$

# The case p(n)=n+2

**Claim 6.15** If there is a PRG $\bar{G}$ with expansion factor $n+1$, then there is a PRG $G$ with expansion factor $n+2$.

# Hybrid distributions

$\bar{G}$ PRG with expansion $n+1$, G corresponding PRG with expansion factor $n+2$

3 distributions on $\{0,1\}^{n+2}$ :

$H_n^0: \quad s_0 \leftarrow \{0,1\}^n, s = G(s_0)$

$H_n^1: \quad s_1' \leftarrow \{0,1\}^n, \sigma_1 \leftarrow \{0,1\}, s = \bar{G}(s_1') \| \sigma_1$

$H_n^2: \quad s \leftarrow \{0,1\}^{n+2}$

# The case p(n)=n+2

**Claim 6.15** If there is a PRG $\bar{G}$ with expansion factor $n+1$, then there is a PRG G with expansion factor $n+2$.

For every ppt D there is a negligible function $\mu(n)$ such that

$$\left|\Pr_{s_2 \leftarrow H_n^0}\left[D(s_2)=1\right]-\Pr_{s_2 \leftarrow H_n^1}\left[D(s_2)=1\right]\right| \le \mu(n).$$

For every ppt D there is a negligible function $\mu(n)$ such that

$$\left|\Pr_{s_2 \leftarrow H_n^1}\left[D(s_2)=1\right]-\Pr_{s_2 \leftarrow H_n^2}\left[D(s_2)=1\right]\right| \le \mu(n).$$

# The case p(n)=n+2

$D$ distinguisher against $G$, construct distinguisher $D'$ against $\overline{G}$ as follows:

$D'$ on input $w \in \{0,1\}^{n+1}$

1. $j \leftarrow \{1,2\}$
2. $\sigma_{j-1} \leftarrow \{0,1\}^{j-1}$
3. $s_j := w \,||\, \sigma_{j-1}$. Run $G$, with input $s_j$ and starting with iteration $i = j+1$, and output $D\left(s_{q(n)}\right)$.

# The case p(n)=n+2

$D'$ on input $w \in \{0,1\}^{n+1}$

1. $j \leftarrow \{1,2\}$
2. $\sigma_j \leftarrow \{0,1\}^{j-1}$
3. $s_j := w \,\|\, \sigma_j$. Run G, with input $s_j$ and starting with iteration $i = j+1$, and output $D\left(s_{q(n)}\right)$.

## Crucial equality

$$\left|\Pr_{w\leftarrow\{0,1\}^{n+1}}\left[D'(w)=1\right]-\Pr_{s\leftarrow\{0,1\}^{n}}\left[D'\left(\bar{G}(s)\right)=1\right]\right|$$

$$= \left|\frac{1}{2}\left(\Pr_{s_2\leftarrow H_n^2}\left[D(s_2)=1\right]-\Pr_{s_2\leftarrow H_n^1}\left[D(s_2)=1\right]\right)\right.$$

$$\left.+\ \frac{1}{2}\left(\Pr_{s_2\leftarrow H_n^1}\left[D(s_2)=1\right]-\Pr_{s_2\leftarrow H_n^0}\left[D(s_2)=1\right]\right)\right|$$

$$= \frac{1}{2}\left|\Pr\left[D(s_2)=1\right]-\Pr\left[D\left(G(s)\right)=1\right]\right|$$

# Hybrid distributions for the general case

$\bar{G}$ PRG with expansion $n+1$, $G$ corresponding PRG with expansion factor $p(n)$, set $q(n)=p(n)-n$.

**Hybrid distribution $H_n^j, 0 \le j \le q(n)$**

1. $s_j \leftarrow \{0,1\}^{n+j}$
2. Run $G$, starting with iteration $j+1$ and with $s_j$ as input.
3. Output $s_{q(n)}$.

# PRFs from PRGs

**Theorem 6.16** If there is a PRG G, then pseudorandom functions exist.

# Truly random functions

$$\text{Func}_n := \left\{ f : \{0,1\}^n \to \{0,1\}^n \right\}$$

$$\left|\text{Func}_n\right| = 2^{n2^n}$$

random function: $f \leftarrow \text{Func}_n$

# Pseudorandom function (PRF)

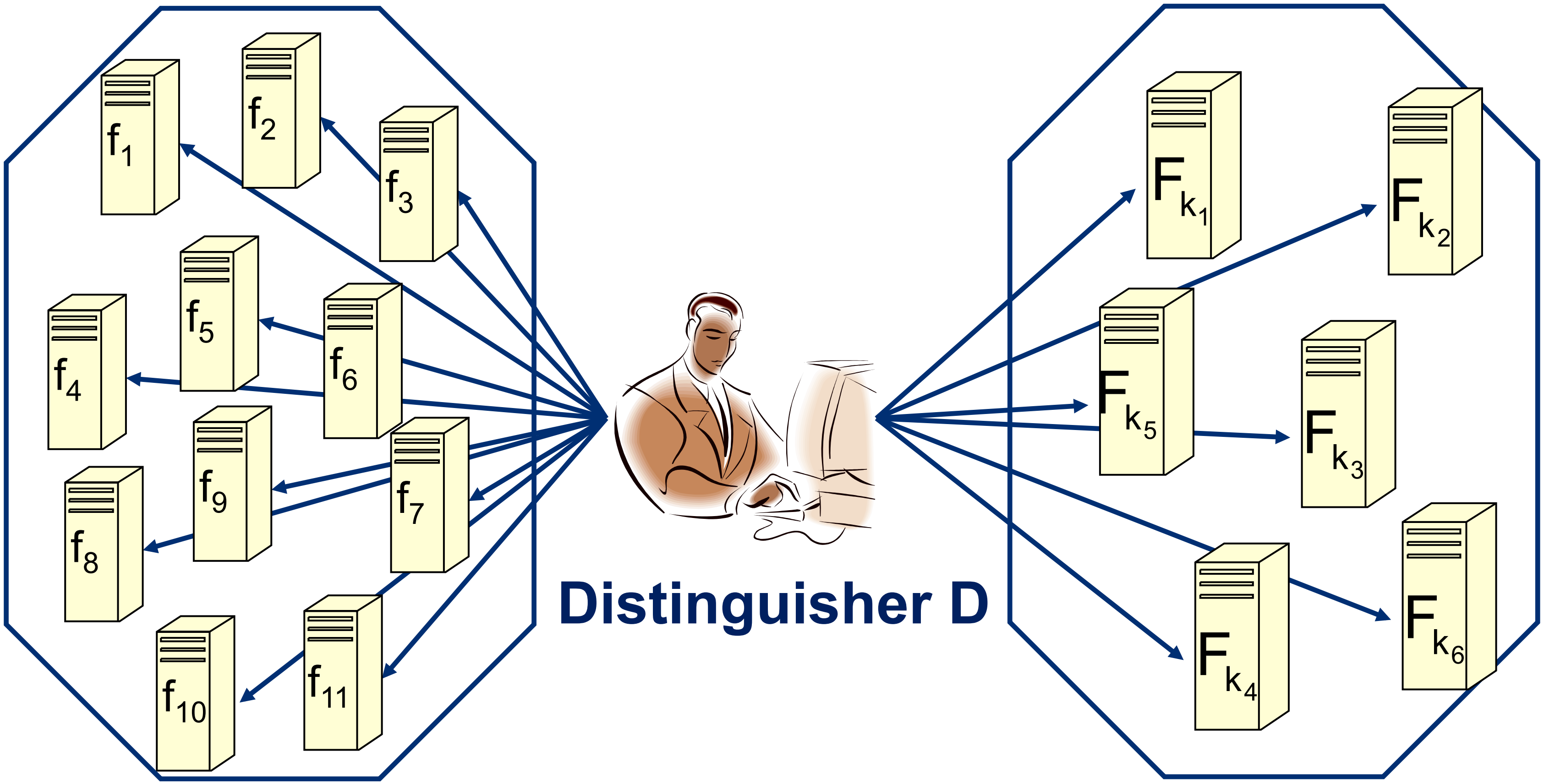
**Definition 3.4 (restated)** Let $F:\{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ be a keyed, efficient and length-preserving function. F is called a pseudorandom function, if for all ppt distinguishers D there is a negligible function $\mu$ such that for all $n \in \mathbb{N}$

$$\left|\Pr\left[D^{F_k(\cdot)}\left(1^n\right)=1\right]-\Pr\left[D^{f(\cdot)}\left(1^n\right)=1\right]\right| \le \mu(n),$$

where $k \leftarrow \{0,1\}^n, f \leftarrow \mathrm{Func}_n$.

$$\mathrm{Func}_n := \left\{f:\{0,1\}^n \to \{0,1\}^n\right\}$$

# Pseudorandom functions




**$\text{Func}_n$**
**with uniform distribution**

$$\mathcal{F}_n = \{F_k(\cdot)\}_{k\in\{0,1\}^n}$$

**with distribution $k \leftarrow \{0,1\}^n$**

# From PRF to cpa-security

**Construction 3.6 (restated)** Let $F:\{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ be a keyed, efficient, and length-preserving function. Define $\Pi_F = (\mathsf{Gen}_F, \mathsf{Enc}_F, \mathsf{Dec}_F)$ as follows:

$\mathsf{Gen}_F$: on input $1^n$: $k \leftarrow \{0,1\}^n$.

$\mathsf{Enc}_F$: on input $k, m \in \{0,1\}^n$, choose $r \leftarrow \{0,1\}^n$ and output $c := (r, m \oplus F_k(r))$.

$\mathsf{Dec}_F$: on input $c = (r, s) \in \{0,1\}^n \times \{0,1\}^n$ and $k \in \{0,1\}^n$ output $m := s \oplus F_k(r)$.

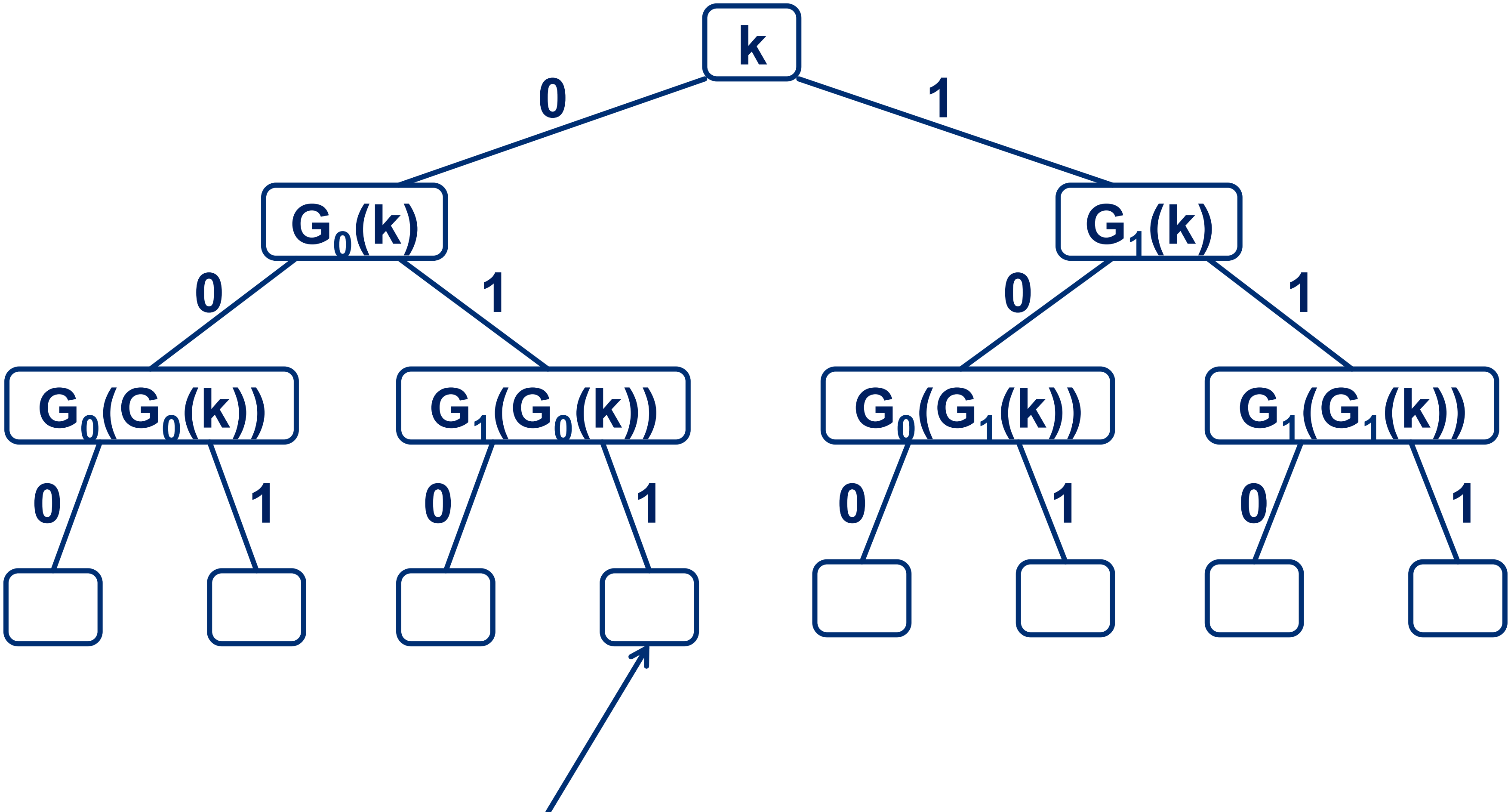
# PRFs from PRGs

**Theorem 6.16** If there is a PRG G, then pseudorandom functions exist.

**Construction 6.17** Let G be a PRG G with expansion factor $p(n) = 2n$. By $G_0(k), G_1(k)$ denote the first and second half of G's output. For every k define the function $F_k$ as follows.

$$\begin{aligned} F_k : \quad \{0,1\}^n &\rightarrow \{0,1\}^n \\ x = x_1 \ldots x_n &\mapsto G_{x_n}\left(\cdots\left(G_{x_2}\left(G_{x_1}(k)\right)\right)\cdots\right) \end{aligned}$$

# PRFs from PRGs




$$F_k(011) = G_1(G_1(G_0(k)))$$

# Hybrid distributions

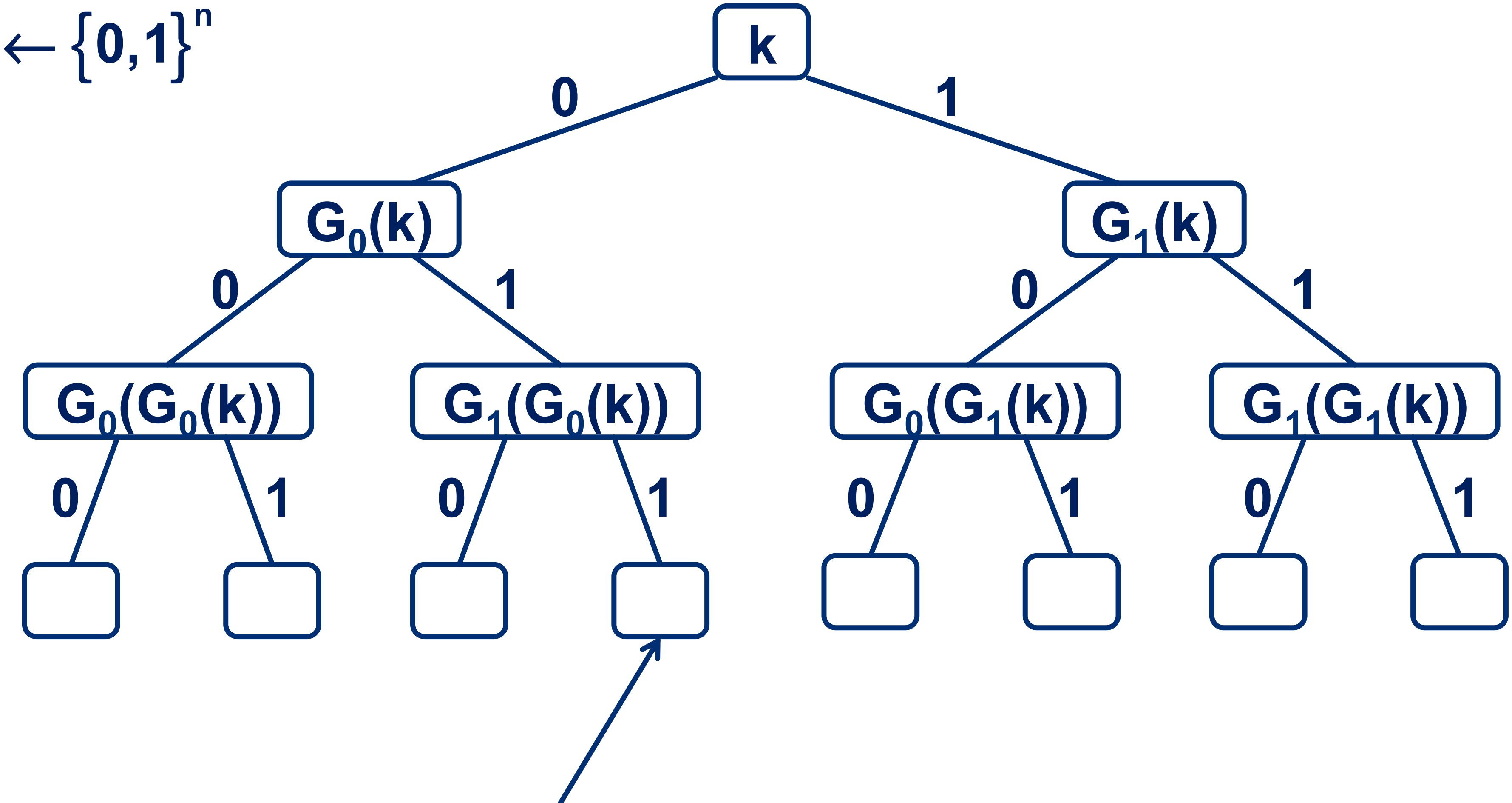
$H_n^i$ distribution on a family of functions $\mathcal{F}_n^i := \{f_s\}_{s \in \{0,1\}^{n2^i}}$

distribution given by $s \leftarrow \{0,1\}^{n2^i}$

# Hybrid distributions

**Hybrid $H_n^0$**

$k \leftarrow \{0,1\}^n$

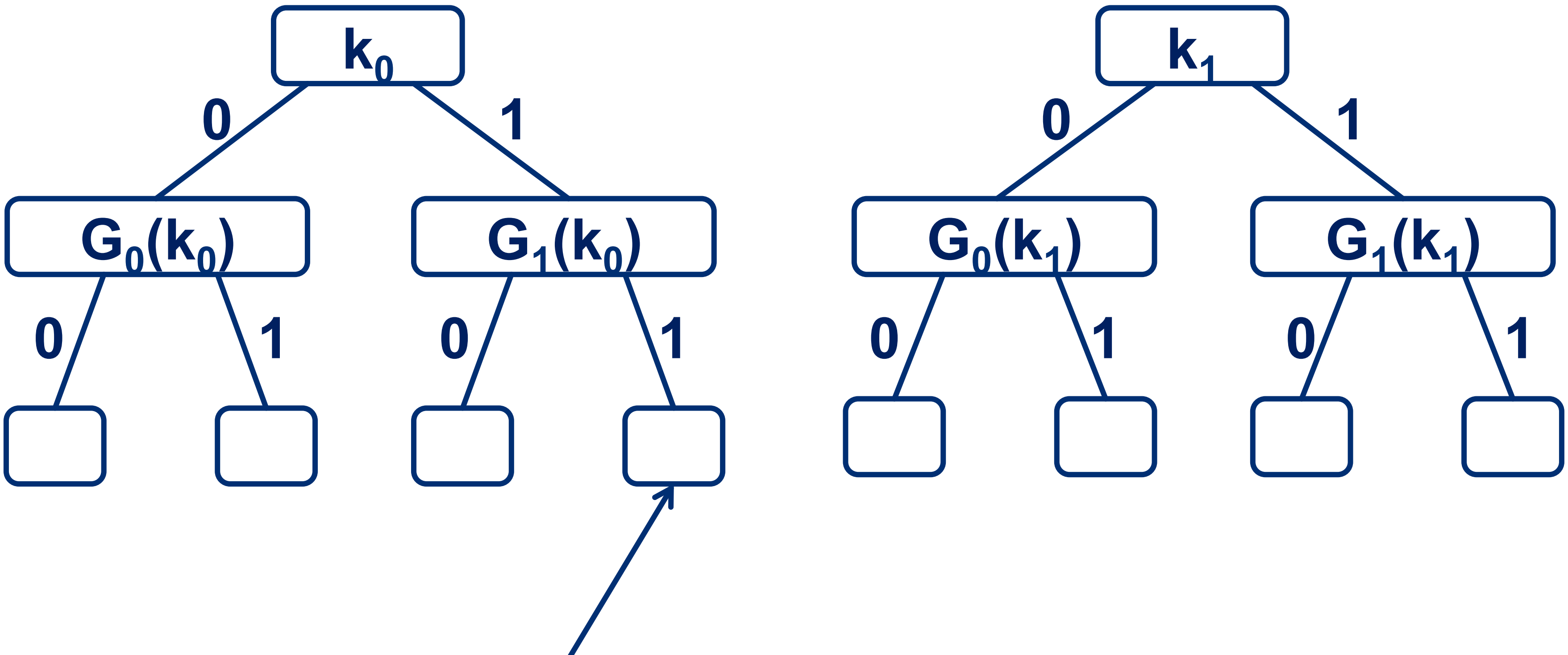



$$F_k(011) = G_1(G_1(G_0(k)))$$

# Hybrid distributions

## Hybrid $H_n^1$

$$k_0 \leftarrow \{0,1\}^n, k_1 \leftarrow \{0,1\}^n$$

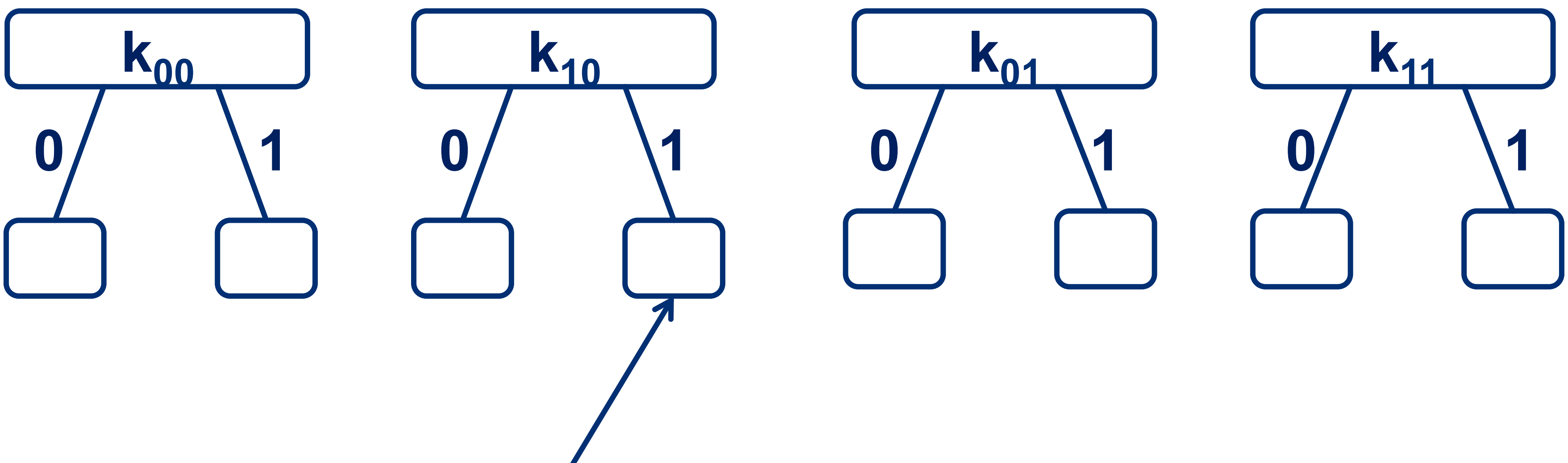



$$F_{k_0 k_1}(011) = G_1(G_1(k_0))$$

# Hybrid distributions

## Hybrid $H_n^2$

$$k_{00} \leftarrow \{0,1\}^n, k_{10} \leftarrow \{0,1\}^n, k_{01} \leftarrow \{0,1\}^n, k_{11} \leftarrow \{0,1\}^n$$

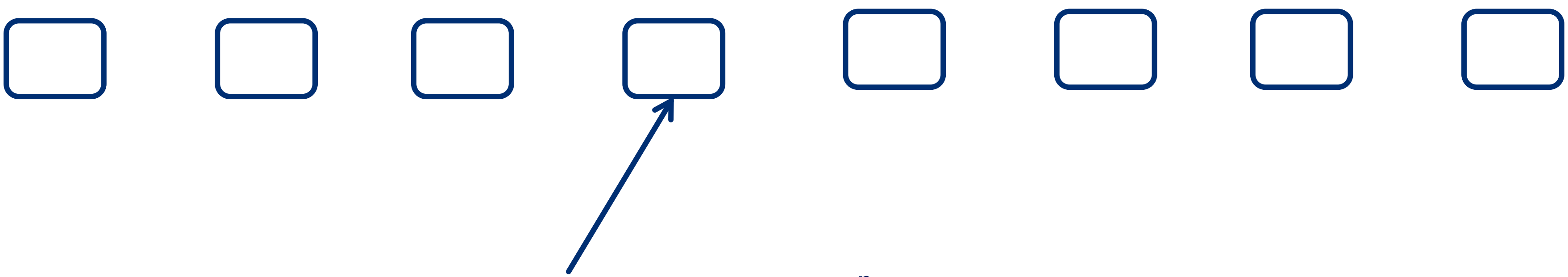



$$F_{k_{00}k_{10}k_{01}k_{11}}(011) = G_1(k_{10})$$

# Hybrid distributions

## Hybrid $H_n^n$

$$k_b \leftarrow \{0,1\}^n, b \in \{0,1\}^n$$



$$F_s(011) = k_{110}, s \in \{0,1\}^{n2^n}$$

# Hybrid distributions

$H_n^i$ distribution on a family of functions $\mathcal{F}_n^i := \{f_s\}_{s \in \{0,1\}^{n2^i}}$

distribution given by $s \leftarrow \{0,1\}^{n2^i}$

- $H_n^0$ pseudorandom function
- $H_n^n$ random function
- $H_n^i$ and $H_n^{i+1}$ differ in one application of G
- Distinguisher for $H_n^0$ and $H_n^n$ leads to distinguisher for $H_n^i$ and $H_n^{i+1}$ for some i.
- Distinguisher for $H_n^i$ and $H_n^{i+1}$ leads to distinguisher for G and uniform distribution.

# Neccessary conditions

**Theorem 6.18** If pseudorandom generators exist, then one-way functions exist.

**Theorem 6.19** If encryption schemes with indistinguishable encryptions against eavesdropping adversaries exist, then one-way functions exist.