# III. Pseudorandom functions & encryption

**Eavesdropping attacks not satisfactory security model**

- no security for multiple encryptions
- does not cover practical attacks

→ **new and stronger security notion: indistinguishable encryption against chosen plaintext attacks**

# The indistinguishability game

**Let A be a probabilistic polynomial time algorithm (ppt).**

**CPA indistinguishability game $\mathrm{PrivK}^{cpa}_{A,\Pi}(n)$**

1. $k \leftarrow \mathrm{Gen}(1^n)$.
2. A receives input $1^n$ and has oracle access to $\mathrm{Enc}_k(\cdot)$.
   Outputs two plaintexts $m_0, m_1 \in \{0,1\}^*$ with $|m_0| = |m_1|$.
3. $b \leftarrow \{0,1\}, c \leftarrow \mathrm{Enc}_k(m_b)$. c given to A.
4. A continues to have oracle access to $\mathrm{Enc}_k(\cdot)$.
   It outputs b'.
5. Output of experiment is 1, if $b = b'$, otherwise output is 0.

Write $\mathrm{PrivK}^{cpa}_{A,\Pi}(n) = 1$, if output is 1. Say A has succeded or A has won.

# Oracle access

Algorithm D has oracle access to function $f: U \to R$, if D

1. can write elements $x \in U$ into special memory cells,
2. in one step receives function value $f(x)$.

Notation Write $D^{f(\cdot)}$ to denote that algorithm D has oracle access to $f(\cdot)$.

# The indistinguishability game

**Definition 3.1** $\Pi = (\text{Gen}, \text{Enc}, \text{Dec})$ has indistinguishable encryptions under chosen plaintext attacks (is cpa-secure) if for every probabilistic polynomial time algorithm A there is a negligible function $\mu : \mathbb{N} \to \mathbb{R}^+$ such that

$$\Pr\left[\text{PrivK}_{A,\Pi}^{\text{cpa}}(n) = 1\right] \le 1/2 + \mu(n).$$

**Observation** A cpa-secure encryption scheme cannot have a deterministic encryption algorithm.

# Multiple messages

**Multiple messages cpa game $\Pr ivK_{A,\Pi}^{mult-cpa}(n)$**

1. $k \leftarrow Gen(1^n)$.
2. A receives input $1^n$ and has oracle access to $Enc_k(\cdot)$.

   A outputs two vectors of messages $M_0 = (m_0^1,\ldots,m_0^t)$,

   $M_1 = (m_1^1,\ldots,m_1^t)$ with $|m_0^i| = |m_1^i|$ for all i.
3. $b \leftarrow \{0,1\}, c_i \leftarrow Enc_k(m_b^i)$. $C = (c_1,\ldots,c_t)$ is given to A.
4. A continues to have oracle access to $Enc_k(\cdot)$.

   A outputs bit b'.
5. Output of experiment is 1, if $b = b'$, otherwise output is 0.

Write $\Pr ivK_{A,\Pi}^{mult-cpa} = 1$, if output is 1. Say A has succeded or A has won.

# CPA-security and multiple messages

**Theorem 3.2** If encryption scheme $\Pi = (\text{Gen}, \text{Enc}, \text{Dec})$ is cpa-secure, then it also has indistinguishable multiple encryption under chosen plaintext attacks.

# CPA-security and blocks of messages

$\Pi = (\text{Gen},\text{Enc},\text{Dec})$ fixed length, $l(n) = 1$.

Define $\Pi' = (\text{Gen}',\text{Enc}',\text{Dec}')$ as follows

$\text{Gen}'$: same as Gen

$\text{Enc}'$: $\text{Enc}'_k(m) = \text{Enc}_k(m_1)\ldots\text{Enc}_k(m_s)$,

$m = m_1\ldots m_s, m_i \in \{0,1\}^{l(n)}$

$\text{Dec}'$: $\text{Dec}'_k(c) = \text{Dec}_k(c_1)\ldots\text{Dec}_k(c_s)$

**Corollary 3.3** If encryption scheme $\Pi = (\text{Gen},\text{Enc},\text{Dec})$ is cpa-secure, then $\Pi' = (\text{Gen}',\text{Enc}',\text{Dec}')$ is cpa-secure.

# Truly random functions

$$\text{Func}_n := \left\{ f : \{0,1\}^n \rightarrow \{0,1\}^n \right\}$$

$$\left|\text{Func}_n\right| = 2^{n2^n}$$

random function: $f \leftarrow \text{Func}_n$

# Keyed functions

$$F: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$$
$$(k,x) \mapsto F(k,x)$$

called keyed function. Write $F(k,x) = F_k(x)$.

- F called length-preserving, if F is only defined for $(x,k) \in \{0,1\}^* \times \{0,1\}^*$ with $|x| = |k|$ and if for all $(x,k)$ $|F_k(x)| = |k| = |x|$.

- F called efficient, if there is a polynomial time algorithm A with $A(k,x) = F_k(x)$ for all $x,k \in \{0,1\}^*$.

- F called permutation, if for every $n \in \mathbb{N}$ and $k \in \{0,1\}^n$ $F_k : \{0,1\}^n \to \{0,1\}^n$ is bijective.

# Oracle access

Algorithm D has oracle access to function $f : U \rightarrow R$, if D

1. can write elements $x \in U$ into special memory cells,
2. in one step receives function value $f(x)$.

Notation Write $D^{f(\cdot)}$ to denote that algorithm D has oracle access to $f(\cdot)$.

# Pseudorandom function (PRF)

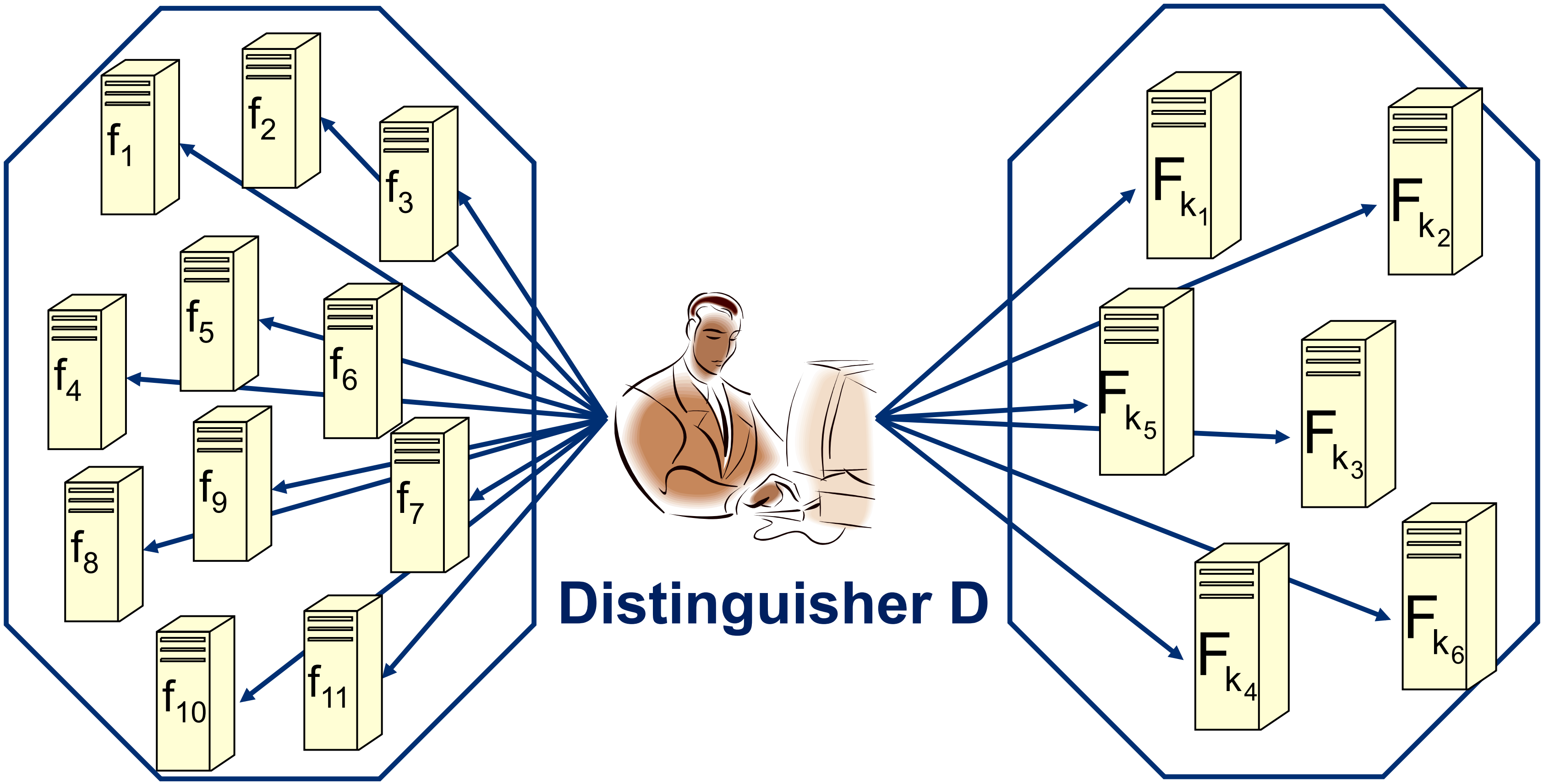
**Definition 3.4** Let $F:\{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ be a keyed, efficient and length-preserving function. F is called a pseudorandom function, if for all ppt distinguishers D there is a negligible function $\mu$ such that for all $n \in \mathbb{N}$

$$\left| \Pr\left[D^{F_k(\cdot)}(1^n) = 1\right] - \Pr\left[D^{f(\cdot)}(1^n) = 1\right] \right| \le \mu(n),$$

where $k \leftarrow \{0,1\}^n, f \leftarrow Func_n$.

$$Func_n := \left\{ f : \{0,1\}^n \to \{0,1\}^n \right\}$$

# Pseudorandom functions




**$\text{Func}_n$**
**with uniform distribution**

$$\mathcal{F}_n = \{F_k(\cdot)\}_{k\in\{0,1\}^n}$$

**with distribution $k \leftarrow \{0,1\}^n$**

# Truly random permutations

$$\mathrm{Perm}_n := \left\{ f : \{0,1\}^n \rightarrow \{0,1\}^n \mid f \text{ is a permutation} \right\}$$

$$\left|\mathrm{Perm}_n\right| = 2^n!$$

random permutation: $f \leftarrow \mathrm{Perm}_n$

# Pseudorandom permutation (PRP)

**Definition 3.5** Let $F: \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}^*$ be a keyed, efficient and length-preserving permutation. F is called a pseudorandom permutation, if for all ppt distinguishers D there is a negligible function $\mu$ such that for all $n \in \mathbb{N}$

$$\left|\Pr\left[D^{F_k(\cdot)}\left(1^n\right)=1\right]-\Pr\left[D^{f(\cdot)}\left(1^n\right)=1\right]\right| \leq \mu(n),$$

where $k \leftarrow \{0,1\}^n, f \leftarrow Perm_n$.

# From PRF to cpa-security

**Construction 3.6** Let $F:\{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}^*$ be a keyed, efficient, and length-preserving function. Define $\Pi_F = \left(\text{Gen}_F, \text{Enc}_F, \text{Dec}_F\right)$ as follows:

$\text{Gen}_F$ : on input $1^n$, choose $k \leftarrow \{0,1\}^n$.

$\text{Enc}_F$ : on input $k, m \in \{0,1\}^n$, choose $r \leftarrow \{0,1\}^n$ and output $c := \left(r, m \oplus F_k(r)\right)$.

$\text{Dec}_F$ : on input $c = (r, s) \in \{0,1\}^n \times \{0,1\}^n$ and $k \in \{0,1\}^n$ output $m := s \oplus F_k(r)$.

# From PRF to cpa-security

**Theorem 3.7** If F is a pseudorandom function, then $\Pi_F$ as defined in Construction 3.6 is cpa-secure.

# From adversaries to distinguishers

D on input $1^n$ and oracle access to $f: \{0,1\}^n \rightarrow \{0,1\}^n$

1. Simulate $A(1^n)$. When A queries for an encryption of $m \in \{0,1\}^n$, answer as follows:
   a) $r \leftarrow \{0,1\}^n$
   b) Query $f(\cdot)$ to obtain $f(r)$ and return $(r, m \oplus f(r))$.
2. When A outputs $m_0, m_1$, choose $b \leftarrow \{0,1\}$, then
   a) $r \leftarrow \{0,1\}^n$
   b) Query $f(\cdot)$ to obtain $f(r)$ and return $c := (r, m_b \oplus f(r))$.
3. Continue to simulate A and answer encryption queries as in 1. Let A's output be $b' \in \{0,1\}$. Output 1, if $b = b'$, otherwise output 0.

# A conceptual scheme

Define $\Pi_{true} = (Gen_{true}, Enc_{true}, Dec_{true})$ as follows:

$Gen_{true}$ : on input $1^n$, choose $f \leftarrow Func_n$.

$Enc_{true}$ : on input $f, m \in \{0,1\}^n$, choose $r \leftarrow \{0,1\}^n$ and output $c := (r, m \oplus f(r))$.

$Dec_{true}$ : on input $c = (r, s) \in \{0,1\}^n \times \{0,1\}^n$ and $f \in Func_n$ output $m := s \oplus f(r)$.

**Remark**

- The scheme is not an encryption scheme, because it is not efficient. It is only used in the proof of Theorem 3.7.

- The CPA indistiguishability experiment can be defined for this scheme.

# From PRF to cpa-security – two basic claims

**Claim 1** For all ppts A

$$\left|\Pr\left[\Pr ivK^{cpa}_{A,\Pi_F}(n)=1\right]-\Pr\left[\Pr ivK^{cpa}_{A,\Pi_{true}}(n)=1\right]\right|$$

$$=\left|\Pr\left[D^{F_k(\cdot)}\left(1^n\right)=1\right]-\Pr\left[D^{f(\cdot)}\left(1^n\right)=1\right]\right|.$$

**Claim 2** Let A be a ppt adversary in $PrivK^{cpa}_{A,\cdot}$ that on input $1^n$ makes at most q(n) oracle queries. Then

$$\left|\Pr\left[Priv^{cpa}_{A,\Pi_{true}}(n)=1\right]\right|\leq\frac{1}{2}+\frac{q(n)}{2^n}.$$

# The CCA indistinguishability game

**CCA indistinguishability game $PrivK^{cca}_{A,\Pi}(n)$**

1. $k \leftarrow Gen(1^n)$
2. A on input $1^n$ has access to encryption algorithm $Enc_k(\cdot)$ and to decryption algorithm $Dec_k(\cdot)$. A outputs 2 messages $m_0, m_1 \in \{0,1\}^*$ of equal length.
3. $b \leftarrow \{0,1\}$, $c \leftarrow Enc_k(m_b)$. c is given to A.
4. $b' \leftarrow A(1^n, c)$, here A has access to encryption algorithm $Enc_k(\cdot)$ and to decryption algorithm $Dec_k(\cdot)$, but query $Dec_k(c)$ is forbidden.
5. Output of experiment is 1, if $b = b'$. Otherwise output is 0.

# CCA-security

**Definition 3.8** $\Pi = (\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ has indistinguishable encryptions under chosen ciphertext attacks (is cca-secure) if for every probabilistic polynomial time algorithm A there is a negligible function $\mu : \mathbb{N} \to \mathbb{R}^{+}$ such that

$$\Pr\left[\mathrm{PrivK}^{\mathrm{cca}}_{A,\Pi}(n) = 1\right] \leq 1/2 + \mu(n).$$

**Observation** cpa-security does not imply cca-security.