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Complexity Theory

SS 2016

Homework 9

Exercise 1 (8 points)**:**
Let

$$CYCLE = \{\langle G\rangle \mid G \text{ is a directed graph that contains a directed cycle}\}.$$

a) Show that $CYCLE \in \mathbf{NL}$

b) Show that $CYCLE$ is **NL**-complete.
Hint: you could either show that $PATH \leq_L CYCLE$ or give a generic reduction from any **NL** language similar to the proof that $PATH$ is **NL**-complete. The latter turns out to be significantly easier because of a convenient property of configuration graphs.

Exercise 2 (10 points)**:**
In this exercise, we examine a witness-based characterization of the class **NL**.
Consider the following definition: A *deterministic Turing machine M with input and witness tape* has three tapes:

- A read-only input tape.
- A special read-only *witness* tape whose head cannot move left (it can only stay in place or move to the right).
- A work tape that works as usual.

We denote the input to M as (x, z), by which we mean that M is started with $\triangleright x\#$ on its input tape and $\triangleright z\#$ on its witness tape. $L(M) := \{(x, z) \mid M \text{ accepts } (x, z)\}$.
The space complexity $s(n)$ of M is the maximum number of cells scanned on the *work* tape when started with any input (x, z) with $|x| = n$ (we require that M halts on all inputs). M is a *log space* TM if its space complexity is $\mathcal{O}(\log(n))$.

a) Show that a language $L \subseteq \{0, 1\}^*$ is in **NL** if and only if there exists a *log space* deterministic Turing machine M with input and witness tape and a polynomial p such that

$$L = \{x \in \{0, 1\}^* \mid \exists z \in \{0, 1\}^{p(|x|)} : (x, z) \in L(M)\}$$

b) Show that if we also allow the witness tape head to move left (but keep the log space restriction), then the class characterized above is **NP** instead of **NL**. More specifically: $L \subseteq \{0, 1\}^*$ is in **NP** if and only if there exists a *log space* deterministic Turing machine M with input and witness tape (where the head is allowed to move left but which is still read-only) and a polynomial p such that

$$L = \{x \in \{0, 1\}^* \mid \exists z \in \{0, 1\}^{p(|x|)} : (x, z) \in L(M)\}$$

Hint: Given a language in **NP**, think about a suitable witness that can be checked in logarithmic space.