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submission due: 30. May 2016, 14:00

Complexity Theory

SS 2016

Homework 6

Exercise 1 (10 points)**:**
Consider the definitions of H and SAT_H from the lecture. Prove Lemma 3.9, i.e.

a) H is well-defined.

b) $H(n)$ can be computed in time $\mathcal{O}(n^3)$.

- As a preparation, show that $\log(n)^{\log\log(n)} = \mathcal{O}(n)$.
- Bound the time needed to compute $H(k)$ for every $k \leq \log(n)$ (argue inductively).
- Bound the time needed to compute SAT solutions for all formulas of length at most $\log(n)$.
- Using the results of the previous two steps, bound the time needed to compute $SAT_H(x)$ for all $x \in \{0,1\}^*$ with $|x| \leq \log(n)$.
- Write down an algorithm computing $H(n)$ assuming you already pre-computed the $SAT_H(x)$ values as above.
- Show that your algorithm takes time $\mathcal{O}(n^3)$.
- Argue that $H(n)$ can be computed in time $\mathcal{O}(n^3)$ (how much time does the preprocessing take?).

You may neglect the time needed for arithmetic operations, variable lookups and any other factors that do not meaningfully contribute to the runtime. You may assume that you can simulate one step of M_i in constant time. Keep your argument short but precise.

Exercise 2 should not *be worked on in groups. Instead,* ***every student must hand in their own individual solution****. On the next homework sheet, one of the exercises will be to correct two randomly assigned solutions from other students. There will be points for both your correct solution on this homework sheet and for correcting peer solutions as part of the next sheet. Make sure that your solution itself does not contain any identifying information (e.g., name, matriculation number). If you hand in your solution to Exercise 2 via email, please mention your name and matriculation number in the email text. If you hand in your solution in the lecture on 30.05. at 14:00, please mark your solution using a removable post-it with your name and matriculation number on it.*
(Note that Exercise 1 can be handed in the usual form (and in groups as usual)).

Exercise 2 (10 points)**:**
Note that the following statements are deliberately simple to prove. For this exercise, we want to focus on precise and meaningful arguments. You should make sure that your proof

structure and each intermediate step can be easily understood. Your proofs should be convincing by themselves, not merely verifiable for people who already know the solutions.

Please prove the following statements iff your matriculation number is *even*:

a) If $\mathbf{P} = \mathbf{NP}$, then every $L \in \mathbf{P} \setminus \{\{0,1\}^*, \emptyset\}$ is $\mathbf{NP}$-complete.

b) Refute: If $L \in \mathbf{P}$ and $L' \subseteq L$, then $L' \in \mathbf{P}$.

c) If $\mathbf{NP} \neq$ co-$\mathbf{NP}$ then $\mathbf{P} \neq \mathbf{NP}$.

d) Let $f : \{0,1\}^* \to \{0,1\}^*$ be a polynomial-space computable function that is length-preserving (i.e. $\forall x \in \{0,1\}^* : \; |f(x)| = |x|$). Then $\mathrm{im}(f) := \{f(x) \mid x \in \{0,1\}^*\} \in \mathbf{PSPACE}$.

e) co-$\mathbf{NPC} = \{L \subseteq \{0,1\}^* \mid \bar{L} \in \mathbf{NPC}\}$ (note that co-$\mathbf{NPC}$ was defined in the lecture).

Please prove the following statements iff your matriculation number is *odd*:

a) $\{0,1\}^*$ is not $\mathbf{NP}$-complete.

b) $\mathbf{NP}$ is closed under concatenation.

c) co-$\mathbf{NP} \subseteq \mathbf{PSPACE}$.

d) Let $f : \{0,1\}^* \to \{0,1\}^*$ be a polynomial-time computable function that is length-preserving (i.e. $\forall x \in \{0,1\}^* : \; |f(x)| = |x|$). Then $\mathrm{im}(f) := \{f(x) \mid x \in \{0,1\}^*\} \in \mathbf{NP}$.

e) co-$\mathbf{NPC} = \{L \subseteq \{0,1\}^* \mid \bar{L} \in \mathbf{NPC}\}$ (note that co-$\mathbf{NPC}$ was defined in the lecture).