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Complexity Theory

SS 2016

Homework 13

Exercise 1 (8 points)**:**
Prove the statement "*if* $\Sigma_k = \Pi_k$ *for some* $k \in \mathbb{N}$*, then the polynomial time hierarchy collapses to its* k*-th level*", i.e.

$$\mathbf{PH} = \Sigma_k \ .$$

Hint: We *suggest* you go through the following steps:

a) Show that if $\Sigma_\ell \subseteq \Sigma_{\ell-1}$ for some $\ell > 1$, then $\Sigma_\ell = \Pi_\ell$.

b) Show by induction that $\Sigma_{\ell+1} \subseteq \Sigma_\ell$ for all $\ell \geq k$ using Theorem 7.4 (twice) and (a).

c) Conclude that $\mathbf{PH} = \Sigma_k$.

Exercise 2 (6 points)**:**
Let $f \in \mathbb{Z}[x]$ be a polynomial with variable x and integer coefficients. We denote the *degree* of f as $\deg(f)$.

Consider the following randomized algorithm $\mathcal{A}$ that checks equality of two given polynomials. On input $f, g \in \mathbb{Z}[x]$, $\mathcal{A}$ does the following:

- Let $d = \max\{\deg(f), \deg(g)\}$.
- Choose a uniformly random integer $z \in \{1, \ldots, 2d\}$.
- If $f(z) = g(z)$ then accept, otherwise reject.

a) Argue that if $f = g$, then $\mathcal{A}$ accepts with probability 1.

b) Argue that if $f \neq g$, then $\mathcal{A}$ accepts with probability at most 1/2.

Hint: Note that $f(z) = g(z) \Leftrightarrow (f - g)(z) = 0$. How many roots (zeros) does $f - g$ have at most?

Exercise 3 (8 points)**:**
Let $L \in \mathbf{RP}$. Show that for any $c \in \mathbb{N}$, there is a polynomial p and a p-balanced NTM N such that

- if $w \notin L$, all computation branches of N reject.
- if $w \in L$, at most $2^{p(|w|)}/2^c$ computation branches of N reject.